# STOCHASTIC OPTIMIZATION USING INTERVAL ANALYSIS, WITH APPLICATIONS TO PORTFOLIO SELECTION[1]

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## Abstract

*In this paper we study a class of optimization problems under uncertainty, with parameters modeled by stochastic random variables. Interval analysis and multiobjective stochastic programming concepts are introduced. Then these two concepts are combined to build a stochastic programming model, with the coefficients of the constraints and the coefficients of the objective function modeled by interval numbers and discrete interval random variables. This model can be used to solve a portfolio optimization problem.*

**Keywords:** *interval analysis, multiobjective stochastic programming, uncertainty, optimization*

## 1. Introduction

The input parameters of the mathematical programming model are not exactly known because relevant data are inexistent or scarce, difficult to obtain or estimate, the system is subject to changes, and so forth, that is, input parameters are uncertain in nature. This type of situations are mainly occurs in real-life decision-making problems. These uncertainties in the input parameters of the model can characterize by interval numbers or random variables with known probability distribution.

The occurrence of randomness in the model parameters can be formulated as stochastic programming (SP) model. SP is widely used in many real-world decision-making problems of management science, engineering, and technology. Also, it has been applied to a wide variety of areas such as, manufacturing product and capacity planning, electrical generation capacity planning, financial planning and control, supply chain management, dairy farm expansion planning, macroeconomic modeling and planning, portfolio selection, traffic management, transportation, telecommunications, and banking

An efficient method known as two-stage stochastic programming (TSP) in which policy scenarios are desired for studying problems with uncertainty. In TSP paradigm, the decision variables are partitioned into two sets. The decision variables which are decided before the actual realization of the uncertain parameters are known as first stage variables. Afterward, once the random events have exhibited themselves, further decision can be made by selecting the values of the second-stage. The formulation of two-stage stochastic programming problems was first introduced by Dantzig [5]. Further it was developed by Barik [2] and Wallcup [15].This article proposes such of stochastic programming

## 2. Interval Analysis

Interval analysis was introduced by Moore [10] . The growing efficiency of interval analysis for solving various real life problems determined the extension of its concepts to the probabilistic case. Thus, the classical concept of random variable was extended to interval random variables, which has the ability to represent not only the randomness character, using the concepts of probability theory, but also imprecision and non-specifficity, using the concepts of interval analysis. The interval analysis based approach provides mathematical models and computational tools for modeling data and for solving optimization problems under uncertainty.

The results presented in this chapter are discussed in more detail in [1, 11],

Let $x^L, x^U$ be real numbers, $x^L \le x^U$.

**Definition 2.1.** An *interval number* is a set defined by:

$$X = [x] = \left\{ x \in \mathbf{R} \,|\, x^L \le x \le x^U;\ x^L, x^U \in \mathbf{R} \right\}.$$

**Remark 2.1.** We will denote by $[x]$ the interval number $[x^L, x^U]$, with $x^L, x^U \in \mathbf{R}$.

We will denote by $\mathbf{IR}$ the set of all interval numbers.

**Definition 2.2.** Let (Ω,K,P) be a probability space and **IR** be the set of the real intervals. An interval random variable [X] is an application [X]: Ω→**R**, defined by:

$[X](\omega) = [X^L(\omega), X^U(\omega)]$, where $X^L$, $X^U$: Ω→**R** are random variables, such that $X^L \le X^U$ almost surely. We say that the interval random variable [X] is a discrete interval random variable if it takes values in a finite subset of the set of the real numbers Otherwise we say that [X] is a continuous interval random variable.

**Definition 2.3.** The *product between the real number a and the interval number* $[x]$ is defined by:

$$a \cdot [x] = \{ a \cdot x | x \in [x] \} = \begin{cases} [a \cdot x^L, a \cdot x^U], & \text{if } a > 0 \\ [a \cdot x^U, a \cdot x^L], & \text{if } a < 0 \\ [0], & \text{if } a = 0 \end{cases}.$$

Let $[x] = [x^L, x^U]$ and $[y] = [y^L, y^U]$ be interval numbers, with $x^L, x^U, y^L, y^U \in \mathbf{R}$.

**Definition 2.4.** The *equality* between interval numbers is defined by:

$[x] = [y]$ if and only if $x^L = y^L$ and $x^U = y^U$ .

**Definition 2.5.**

The *summation* of two interval numbers is defined by: $[x] + [y] = [x^L + y^L, x^U + y^U]$.

The *subtraction* of two interval numbers is defined by: $[x] - [y] = [x^L - y^U, x^U - y^L]$.

The *product between* two interval numbers is defined by:

$$[x] \cdot [y] = \left[ \min\{x^L y^L, x^L y^U, x^U y^L, x^L y^U\}, \max\{x^L y^L, x^L y^U, x^U y^L, x^L y^U\} \right]$$

If $0 \notin [y]$, then $\frac{1}{[y]}$ is defined by: $\frac{1}{[y]} = \left[\frac{1}{y^U}, \frac{1}{y^L},\right]$

If $0 \notin [y]$, then the division between two interval numbers is defined by:

$$\frac{[x]}{[y]} = \left[\min\left\{\frac{x^L}{y^U}, \frac{x^U}{y^L}\right\}, \max\left\{\frac{x^L}{y^U}, \frac{x^U}{y^L}\right\}\right].$$

**Definition 2.6.** $[x] \leq [y]$ if f $x^L \leq y^L$ and $x^U \leq y^U$;

## 3. Stochastic Programming

### 3.1. Multiobjective Stochastic Programming

In stochastic or probabilistic programming some or all of the parameters of the optimization problem are described by stochastic or random variables rather than by deterministic quantities. In recent years, multiobjective stochastic programming problems have become increasingly important in scientifically based decision making involved in practical problem arising in economic, industry, healthcare, transportation, agriculture, military purposes, and technology. Mathematically, a multiobjective stochastic programming problem can be stated as follows:

$$\begin{aligned} \max \quad & z^t = \sum_{j=1}^{n} c_j^t ]x_j, \; t = 1,2,\ldots, \; T \\ \text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i, \; i = 1,2,\ldots, \; m_1 \\ & \sum_{j=1}^{n} d_{ij} x_j \leq b_{m_1+i}, \; i = 1,2,\ldots, \; m_2 \\ & x_j \geq 0, \; j = 1,2,\ldots, \; n \end{aligned} \tag{3.1}$$

where the parameters $a_{ij}, i = 1,2,\ldots,m_1, j = 1,2,\ldots,n$ and $b_i, i = 1,2,\ldots,m_1$ are discrete random variables with known probability distributions. The rest of the parameters $c_j^t, j = 1,2,\ldots,n, t = 1,2,\ldots,T$, $d_{ij}, i = 1,2,\ldots,m_2, j = 1,2,\ldots,n$ and $b_{m_1+i}, i = 1,2,\ldots,m_2$ are considered as known intervals.

### 3.2. Multiobjective Two-Stage Stochastic Programming

In two-stage stochastic programming (TSP), decision variables are divided into two subsets:

(1) a group of variables determined before the realizations of random events are known as first stage decision variables, and

(2) another group of variables known as recourse variables which are determined after knowing the realized values of the random events.

A general model of TSP with simple recourse can be formulated as follows [3,9,15]:

$$\max \ \bar{z} = \sum_{j=1}^{n} c_j x_j - E\left( \sum_{i=1}^{m_1} q_i |y_i| \right)$$

$$\text{subject to } y_i = b_i - \sum_{j=1}^{n} a_{ij} x_j ,\ i = 1,2,...,\ m_1$$

$$\sum_{j=1}^{n} d_{ij} x_j \le b_{m_1+i} ,\ i = 1,2,...,\ m_2 \tag{3.2}$$

$$x_j \ge 0,\ j = 1,2,...,\ n$$

$$y_i \ge 0,\ i = 1,2,...,\ m_1$$

where $x_j, j = 1,2,...,n$ and $y_i, i = 1,2,...,m_1$ are the first stage decision variables and second stage decision variables respectively.

Further, $q_i, i = 1,2,...,m_1$ are defined as the penalty costs associated with the discrepancies between $\sum_{j=1}^{n} a_{ij} x_j$ and $b_i$ and $E$ is used to represent the expected value of a random variable.

Multiobjective optimization problems appear in most of the real life decision making problems. Thus, a general model of multiobjective stochastic programming model (3.1) can be stated as follows:

$$\max \ z^t = \sum_{j=1}^{n} c_j^t x_j - E\left( \sum_{i=1}^{m_1} q_i^t |y_i| \right), \ t = 1,2,...,T$$

$$\text{subject to } y_i = b_i - \sum_{j=1}^{n} a_{ij} x_j ,\ i = 1,2,...,\ m_1$$

$$\sum_{j=1}^{n} d_{ij} x_j \le b_{m_1+i} ,\ i = 1,2,...,\ m_2 \tag{3.3}$$

$$x_j \ge 0,\ j = 1,2,...,\ n$$

$$y_i \ge 0,\ i = 1,2,...,\ m_1$$

## 4. Random Interval Multiobjective Two-Stage Stochastic Programming

Optimization model incorporating some of the input parameters as interval random variables is modeled as random interval multiobjective two-stage stochastic programming (RIMTSP) to handle the uncertainties within TSP optimization platform with simple recourse. Mathematically, it can be presented as follows:

$$\max\ \hat{z}^t = \sum_{j=1}^{n}\left[c_j^t\right]x_j - E\left(\sum_{i=1}^{m_1} q_i^t\left|y_i\right|\right),\ t = 1,2,...,T$$

$$\text{subject to } y_i = [B_i] - \sum_{j=1}^{n}[A_{ij}]x_j,\ i = 1,2,...,\ m_1$$

$$\sum_{j=1}^{n}\left[d_{ij}\right]x_j \le \left[b_{m_1+i}\right],\ i = 1,2,...,\ m_2 \tag{4.1}$$

$$x_j \ge 0,\ j = 1,2,...,\ n$$

$$y_i \ge 0,\ i = 1,2,...,\ m_1$$

where $x_j,\ j = 1,2,...,n$ and $y_i, i = 1,2,...,m_1$ are the first stage decision variables and second stage decision variables respectively. Further, $\left[c_j^t\right],\ j = 1,2,...,n,\ t = 1,2,...,T$ are the costs associated with the first stage decision variables and $q_i^t$, $i = 1,2,...,m_1$, $t = 1,2,...,T$ are the penalty costs associated with the discrepancie between $\sum_{j=1}^{n}[A_{ij}]x_j$ and $[B_i]$ of the *k*th objective function. The left hand side parameter $[A_{ij}]$ and the right hand side parameter $[B_i]$ are discrete interval random variables, with known probability distributions and between *E* is used to represent the expected value associated with interval random variables.

## 5. Conclusions

The new approach based on interval analysis provides mathematical models and computational tools for modeling the imprecision of financial data and for solving decision making problems under uncertainty. This article have proposed a random interval multiobjective two-stage stochastic programming

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