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WEIGHTED SHANKER DISTRIBUTION AND ITS APPLICATIONS TO MODEL LIFETIME DATA

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Abstract

A two-parameter weighted Shanker distribution which includes one parameter Shanker distribution introduced by Shanker (2015) as a particular case has been proposed for modeling lifetime data. Structural properties of the proposed distribution including moments and moment related measures, hazard rate function, mean residual life function and stochastic orderings have been discussed. The estimation of its parameters has been discussed using maximum likelihood estimation and statistical inferences based on estimated values of parameters have been presented. The goodness of fit of the proposed distribution have been discussed with two real lifetime data sets and the fit has been compared with one parameter exponential, Lindley and Shanker distributions, and two-parameter weighted Lindley distribution and the fit in all data sets is found to be quite satisfactory.

Key words: Shanker distribution; Lindley distribution; moments; hazard rate function; means residual life function; stochastic ordering; maximum likelihood estimation; goodness of fit

1. Introduction

The natural populations of human, wildlife, insect, plant, fish etc do not follows well defined sampling structures and, therefore, recorded observations of individuals from these populations are biased and will not have the original distribution unless every observation is given an equal chance of being recorded. The weighted distribution gives a unified approach to model biased data from natural populations. The concept of weighted distributions to model ascertainment biases have been introduced by Fisher (1934) which were later formulized by Rao (1965) in a unifying theory for problems where the observations fall in non-experimental, non-replicated and non-random. When an investigator records an observation in the nature according to certain stochastic model, the distribution of the recorded observation will not have the original distribution unless every observation is given an equal chance of being recorded. For example, suppose the original observation x_0 comes from a distribution having probability density function (p.d.f.) $f_0\left(x;\theta_1\right)$, where θ_1 may be a parameter vector, and observation x is recorded according to a probability re-weighted by a weight function $w(x;\theta_2)>0$, θ_2 being a new parameter vector, then x comes from a distribution having p.d.f.

$$f(x;\theta_1,\theta_2)=A\,w(x;\theta_2)\,f_0(x;\theta_1) \tag{1.1}$$

where A is a normalizing constant. It should be noted that such types of distributions are known as weighted distributions. The weighted distributions with weight function $w(x;\theta_2)=x$ are called length-biased distribution or simple size-biased distribution. Patil and Rao (1977, 1978) have examined some general probability models leading to weighted probability distributions, discussed their applications and showed the occurrence of $w(x;\theta_2)=x$ in a natural way in problems relating to sampling.

In distribution theory, the study of weighted distribution is useful because it provides a new understanding of the existing standard probability distributions and it provides methods for extending existing standard probability distributions for modeling lifetime data due to additional parameter which creates flexibility in their nature. Weighted distributions occur in modeling clustered sampling, heterogeneity, and extraneous variation in the data set.

Shanker (2015) has introduced a lifetime distribution named Shanker distribution for modeling lifetime data having p.d.f

$$f_0(x;\theta)=\frac{\theta^2}{\theta^2+1}(\theta+x)e^{-\theta x}\quad ;x>0,\ \theta>0 \tag{1.2}$$

Shanker (2015) has derived and discussed its statistical and mathematical properties including its shapes, moments, skewness, kurtosis, hazard rate function, mean residual life function, stochastic ordering, mean deviations, order statistics, Bonferroni and Lorenz curves, Renyi entropy measure, and stress-strength reliability. Shanker (2015) has discussed the estimation of its parameter using both maximum likelihood estimation and method of moments and its detailed applications for modeling real lifetime data from biomedical sciences and engineering. Shanker (2016) has obtained the Poisson mixture of Shanker distribution named Poisson-Shanker distribution (PSD) and discussed its various mathematical and statistical properties, estimation of parameter and applications for various count data-sets. Shanker et al (2017) have detailed study on applications of PSD for modeling count data from different fields of knowledge and shown that in majority of data sets PSD gives better fit than both Poisson distribution and Poisson-Lindley distribution (PLD).

Ghitany et al (2011) introduced a two-parameter weighted Lindley distribution (WLD) with parameters θ and α defined by its p.d.f.

$$f(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{\theta+\alpha}\frac{x^{\alpha-1}}{\Gamma(\alpha+1)}(1+x)e^{-\theta x}\ ;x>0,\theta>0,\alpha>0 \tag{1.3}$$

It can be easily verified that the Lindley distribution introduced by Lindley (1958) having p.d.f.

$$f(x;\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}\ ;x>0,\theta>0 \tag{1.4}$$

is a particular case of (1.3) for $\alpha=1$. Ghitany *et al* (2008) have detailed study regarding its statistical and mathematical properties, estimation of parameter and application. The p.d.f of three-parameter generalized Lindley distribution (GLD) introduced by Zakerzadeh and Dolati (2009) having parameters $\alpha,\beta,\text{and }\theta$ is given by

$$f\left(x;\alpha,\beta,\theta\right)=\frac{\theta^{\alpha+1}}{\left(\beta+\theta\right)}\frac{x^{\alpha-1}}{\Gamma\left(\alpha+1\right)}\left(\alpha+\beta x\right)e^{-\theta x};\; x>0,\alpha>0,\beta>0,\theta>0 \quad (1.5)$$

where

$$\Gamma\left(\alpha\right)=\int_0^\infty e^{-y}\, y^{\alpha-1}\,dy;\; \alpha>0$$

is the complete gamma function.

It can be easily verified that the gamma distribution, the Lindley (1958) distribution and the exponential distribution are particular cases of (1.1) for $\left(\beta=0\right)$, $\left(\alpha=\beta=1\right)$ and $\left(\alpha=1,\beta=0\right)$, respectively. Shanker (2016) have comparative study of GLD and generalized gamma distribution (GGD) and found that in majority of lifetime data from engineering and biomedical science, GGD gives better fit than GLD. Shanker *et al* (2015) have comparative study of exponential and Lindley distribution for modeling of lifetime data and concluded that these two distributions are competing. Sankaran (1970) obtained discrete Poisson-Lindley distribution by mixing Poisson distribution with Lindley distribution and discussed estimation of parameter and applications for count data. Shanker and Hagos (2015) have detailed study about applications of Poisson-Lindley for modeling count data in Biological sciences. Shanker and Mishra (2013 a) has introduced a two-parameter quasi Lindley distribution (QLD) and studied its various properties, estimation of parameters and applications and showed that QLD gives better fit that both exponential and Lindley distributions. Shanker *et al* (2016) have discussed interesting properties of QLD and its detailed applications for modeling lifetime data from engineering and biomedical sciences and showed its superiority over some one parameter lifetime distributions. Shanker *et al* ((2013) has obtained size-biased quasi Poisson-Lindley distribution (SBQPLD), discussed its mathematical and statistical properties, estimation of parameters, and applications for zero-truncated data and concluded that SBQPLD is an important model.

There are many situations in the modeling of real lifetime data where the Lindley (1958) and Shanker (2015) distributions may not be suitable from a theoretical or applied point of view. In the present paper, a two-parameter weighted Shanker distribution has been introduced which includes Shanker distribution as particular case. Its various properties including properties based on moments, hazard rate function, mean residual life function and stochastic ordering have been discussed. The estimation of its parameters has been discussed using maximum likelihood estimation. The goodness of fit of the proposed distribution has been discussed along with one parameter exponential, Lindley and Shanker distributions and two-parameter weighted Lindley distribution.

2. Weighted Shanker distribution

The p.d.f. of a two – parameter weighted Shanker distribution (WSD) can be obtained as

$$f\left(x;\theta,\alpha\right)=A\,x^{\alpha-1}\,f_0\left(x;\theta\right);x>0,\;\theta>0,\;\alpha>0 \quad (2.1)$$

where A is a normalizing constant and $f_0\left(x;\theta\right)$ is the p.d.f. of Shanker distribution given in (1.2). Thus the p.d.f. of weighted Shanker distribution can be obtained as

$$f\left(x;\theta,\alpha\right)=\frac{\theta^{\alpha+1}}{\left(\theta^2+\alpha\right)}\frac{x^{\alpha-1}}{\Gamma\left(\alpha\right)}\left(\theta+x\right)e^{-\theta x}\;;x>0,\;\theta>0,\;\alpha>0 \quad (2.2)$$

where

$$\Gamma(\alpha)=\int_{0}^{\infty} e^{-y}\, y^{\alpha-1}\, dy\,;\, y>0,\ \alpha>0$$

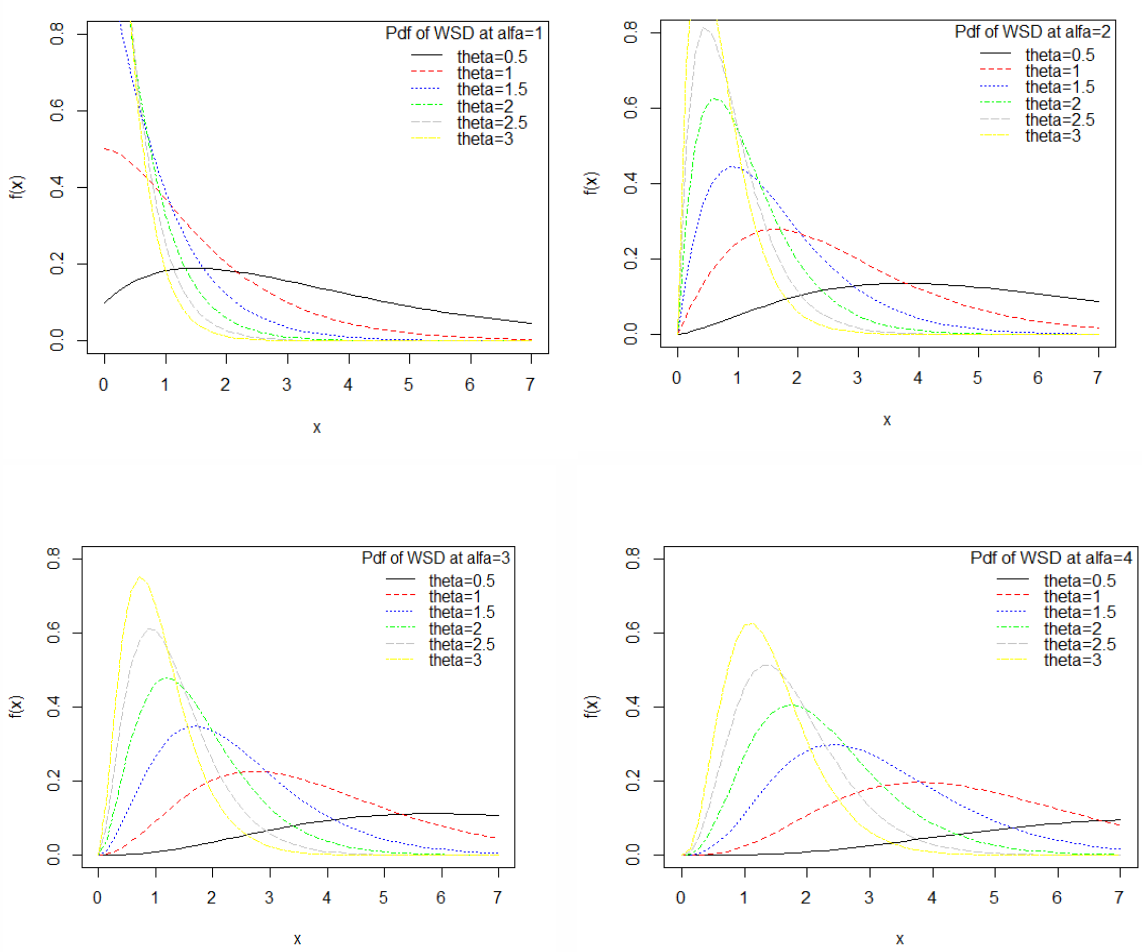
is the complete gamma function. In WSD (2.2), α is the shape parameter and θ is the scale parameter. It can be easily shown that at $\alpha=1$, WSD reduces to Shanker distribution (1.2). Further, p.d.f. (2.2) can be expressed as a two-component mixture of gamma (α,θ) and gamma $(\alpha+1,\theta)$ distributions. We have

$$f\left(x;\theta,\alpha\right)=p\, f_1\left(x;\theta,\alpha\right)+\left(1-p\right) f_2\left(x;\theta,\alpha+1\right), \tag{2.3}$$

where

$$p=\frac{\theta^2}{\theta^2+\alpha},\quad f_1\left(x;\theta,\alpha\right)=\frac{\theta^{\alpha}}{\Gamma(\alpha)}\, e^{-\theta x}x^{\alpha-1}, \text{ and } f_2\left(x;\theta,\alpha+1\right)=\frac{\theta^{\alpha+1}}{\Gamma(\alpha+1)}\, e^{-\theta x}x^{\alpha+1-1}.$$

To study the nature and behavior of WSD for varying values of parameters (θ,α), various graphs have been presented in figure 1.



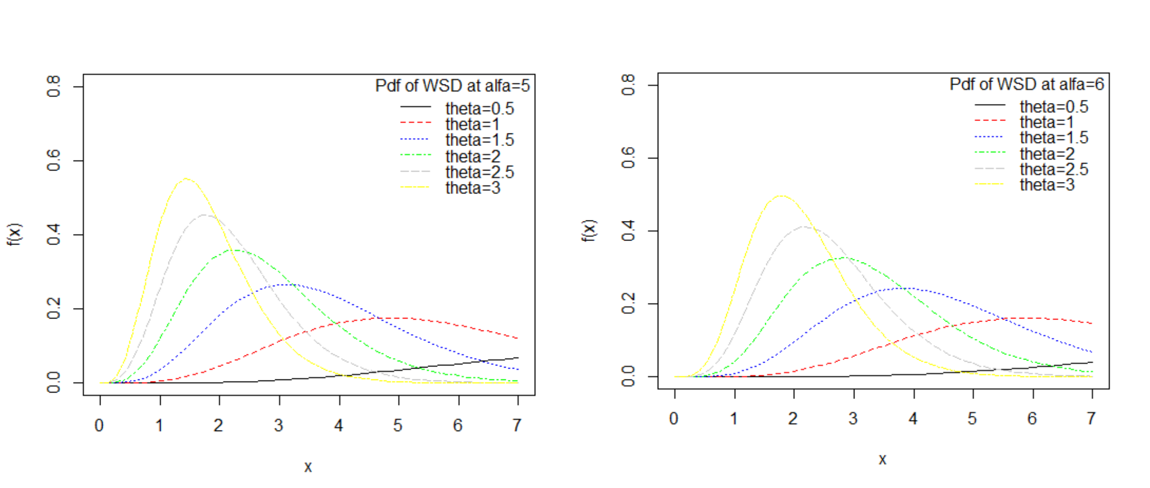


Figure 1. Graphs of the pdf of WSD for varying values of parameters

3. Moments and associated measures

The r th moment about origin of WSD (2.2), using mixture representation (2.3), can be obtained as

$$\mu_r' = E\left(X^r\right) = p\int_0^\infty x^r f_1\left(x;\theta,\alpha\right)dx + \left(1-p\right)\int_0^\infty x^r f_2\left(x;\theta,\alpha+1\right)dx$$

$$= \frac{\left(\theta^2+\alpha+r\right)\Gamma\left(\alpha+r\right)}{\theta^r\left(\theta^2+\alpha\right)\Gamma\left(\alpha\right)};\ r=1,2,3,... \qquad (3.1)$$

Substituting $r=1,2,3,$ and 4 in (3.1), the first four moments about origin of WSD are obtained as

$$\mu_1' = \frac{\alpha\left(\theta^2+\alpha+1\right)}{\theta\left(\theta^2+\alpha\right)}$$

$$\mu_2' = \frac{\alpha\left(\alpha+1\right)\left(\theta^2+\alpha+2\right)}{\theta^2\left(\theta^2+\alpha\right)}$$

$$\mu_3' = \frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\left(\theta^2+\alpha+3\right)}{\theta^3\left(\theta^2+\alpha\right)}$$

$$\mu_4' = \frac{\alpha\left(\alpha+1\right)\left(\alpha+2\right)\left(\alpha+3\right)\left(\theta^2+\alpha+4\right)}{\theta^4\left(\theta^2+\alpha\right)}$$

Again using relationship between central moments and moments about origin, the central moments about the mean of WSD are obtained as

$$\mu_2 = \frac{\alpha\left[\theta^4+2\left(\alpha+1\right)\theta^2+\alpha\left(\alpha+1\right)\right]}{\theta^2\left(\theta^2+\alpha\right)^2}$$

$$\mu_3 = \frac{2\alpha\left[\theta^6 + 3(\alpha+1)\theta^4 + 3\alpha(\alpha+1)\theta^2 + \alpha^2(\alpha+1)\right]}{\theta^3\left(\theta^2+\alpha\right)^3}$$

$$\mu_4 = \frac{3\alpha\begin{bmatrix}(\alpha+2)\theta^8 + 4\left(\alpha^2+3\alpha+2\right)\theta^6 + 2\alpha\left(3\alpha^2+11\alpha+8\right)\theta^4 \\ +4\alpha^2\left(\alpha^2+4\alpha+3\right)\theta^2 + \alpha^3\left(\alpha^2+4\alpha+3\right)\end{bmatrix}}{\theta^4\left(\theta^2+\alpha\right)^4}$$

The expressions for coefficient of variation (C.V.) coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$, and index of dispersion $\left(\gamma\right)$ of WSD are thus obtained as

$$C.V. = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\alpha\left[\theta^4 + 2(\alpha+1)\theta^2 + \alpha(\alpha+1)\right]}}{\alpha\left(\theta^2+\alpha+1\right)}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{2\alpha\left[\theta^6 + 3(\alpha+1)\theta^4 + 3\alpha(\alpha+1)\theta^2 + \alpha^2(\alpha+1)\right]}{\left[\alpha\left\{\theta^4 + 2(\alpha+1)\theta^2 + \alpha(\alpha+1)\right\}\right]^{3/2}}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\alpha\begin{bmatrix}(\alpha+2)\theta^8 + 4\left(\alpha^2+3\alpha+2\right)\theta^6 + 2\alpha\left(3\alpha^2+11\alpha+8\right)\theta^4 \\ +4\alpha^2\left(\alpha^2+4\alpha+3\right)\theta^2 + \alpha^3\left(\alpha^2+4\alpha+3\right)\end{bmatrix}}{\left[\alpha\left\{\theta^4 + 2(\alpha+1)\theta^2 + \alpha(\alpha+1)\right\}\right]^2}$$

$$\gamma = \frac{\sigma^2}{\mu_1'} = \frac{\theta^4 + 2(\alpha+1)\theta^2 + \alpha(\alpha+1)}{\theta\left(\theta^2+\alpha\right)\left(\theta^2+\alpha+1\right)}$$

To study the nature of coefficient of variation (C.V.), coefficient of skewness $\left(\sqrt{\beta_1}\right)$, coefficient of kurtosis $\left(\beta_2\right)$ and index of dispersion $\left(\gamma\right)$ of WSD, their values have been computed for varying values of parameters θ and α and presented in tables 1, 2, 3 and 4.

Table 1. Coefficient of variation (C.V) of WSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	1.010153	1.232883	1.378705	1.40456	1.410746	1.412702
1	0.777778	0.881917	0.971825	0.991701	0.996909	0.998627
2	0.598321	0.637377	0.685119	0.699702	0.704163	0.705753
3	0.509427	0.52915	0.559017	0.570477	0.574456	0.575976
4	0.452381	0.46398	0.484322	0.493581	0.497157	0.498609
5	0.411437	0.418939	0.43359	0.44121	0.444433	0.445815

Table 1 demonstrate that for a given value of $\alpha(\theta)$, C.V increases (decreases) as the value of $\theta(\alpha)$ increases.

Table 2. Coefficient of Skewness $\left(\sqrt{b_1}\right)$ of WSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	1.787566	2.219872	2.667337	2.778657	2.809461	2.819903
1	1.451895	1.619848	1.876396	1.958191	1.983357	1.992342
2	1.161334	1.206869	1.323613	1.378499	1.398831	1.406819
3	1.002109	1.019904	1.083251	1.122849	1.140004	1.147347
4	0.89532	0.90382	0.941812	0.971333	0.985946	0.992705
5	0.816945	0.821565	0.845981	0.868511	0.881023	0.887231

It is clear from table 2 that for a given value of $\theta(\alpha)$, $\sqrt{\beta_1}$ decreases (increases) as the value of $\alpha(\theta)$ increases.

Table 3. Coefficient of Kurtosis (b_2) of WSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	7.6272	10.01385	13.42662	14.4791	14.79528	14.90652
1	6.1212	6.795918	8.15917	8.694089	8.873963	8.940946
2	5.016461	5.147929	5.574669	5.818923	5.918823	5.960102
3	4.504435	4.546485	4.731111	4.870413	4.937506	4.967923
4	4.20166	4.219043	4.313019	4.400496	4.448868	4.472697
5	4.000755	4.009194	4.062291	4.120585	4.15686	4.176119

It is obvious from table 3 that for a given value of $\theta(\alpha)$, β_2 decreases (increases) as the value of $\alpha(\theta)$ increases.

Table 4. Index of dispersion (γ) of WSD for varying values of θ and α

α \ θ	0.5	1	2	3	4	5
0.5	2.380952	1.266667	0.580808	0.363409	0.263853	0.207399
1	2.177778	1.166667	0.566667	0.360606	0.263072	0.207123
2	2.068376	1.083333	0.547619	0.356061	0.261696	0.206614
3	2.036199	1.05	0.535714	0.352564	0.260526	0.206158
4	2.022409	1.033333	0.527778	0.349817	0.259524	0.205747
5	2.015238	1.02381	0.522222	0.347619	0.258658	0.205376

It is clear from table 4 that for a given value of $\theta(\alpha)$, γ decreases as the value of $\alpha(\theta)$ increases.

4. Reliability measures

There are two important reliability measures of a distribution namely, the hazard arte function and the mean residual life function. In this section, the hazard rate function and the mean residual life function of WSD have been computed and their nature have been explained graphically

4.1 Hazard Rate Function

The survival (reliability) function of WSD, using the mixture representation (2.3), can be obtained as

$$S(x)=P(X>x)=p\int_x^\infty f_1(y;\theta,\alpha)dy+(1-p)\int_x^\infty f_2(y;\theta,\alpha+1)dy$$

$$=\frac{(\theta^2+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^\alpha e^{-\theta x}}{(\theta^2+\alpha)\Gamma(\alpha)} \quad (4.1.1)$$

where

$$\Gamma(\alpha,z)=\int_z^\infty e^{-y}y^{\alpha-1}dy \ \ ;y\geq 0,\alpha>0 \quad (4.1.2)$$

is the upper incomplete gamma function.

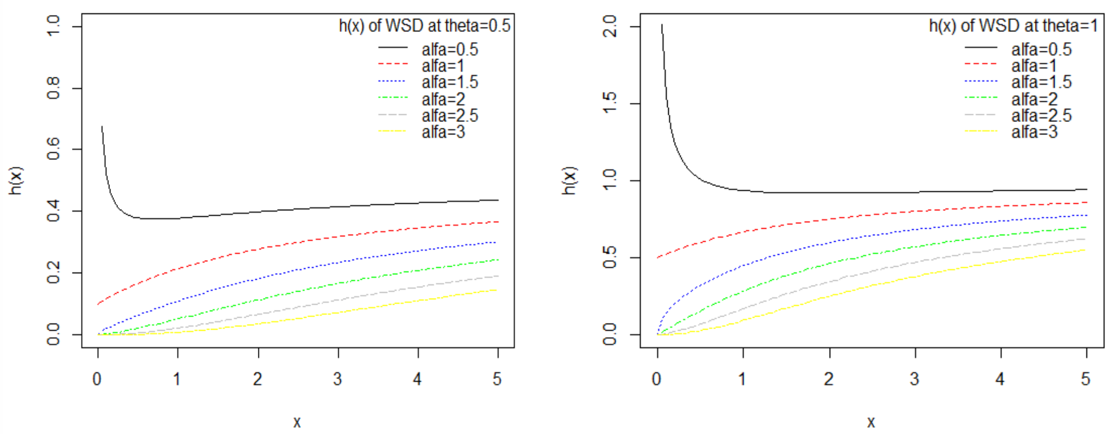
Thus, the hazard (or failure rate) function, $h(x)$ of WSD can be obtained as

$$h(x)=\frac{f(x)}{S(x)}=\frac{\theta^{\alpha+1}x^{\alpha-1}(\theta+x)e^{-\theta x}}{(\theta^2+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^\alpha e^{-\theta x}} \ ;x>0,\theta>0,\alpha>0 \quad (4.1.3)$$

The behavior of $h(x)$ of WSD at $x=0$ and $x=\infty$, respectively, are given by

$$h(0)=f(0)=\begin{cases}\infty & ,if\,\alpha<1\\ \dfrac{\theta^2}{\theta^2+1} & ,if\,\alpha=1\\ 0 & ,if\,\alpha>1\end{cases} \quad , \qquad h(\infty)=\theta$$

The shapes of the hazard rate function, $h(x)$ of the WSD for varying values of parameters (θ,α) are shown in the figure 2. It is obvious from the graphs of $h(x)$ that it is decreasing, increasing, upside bathtub or downside bathtub.



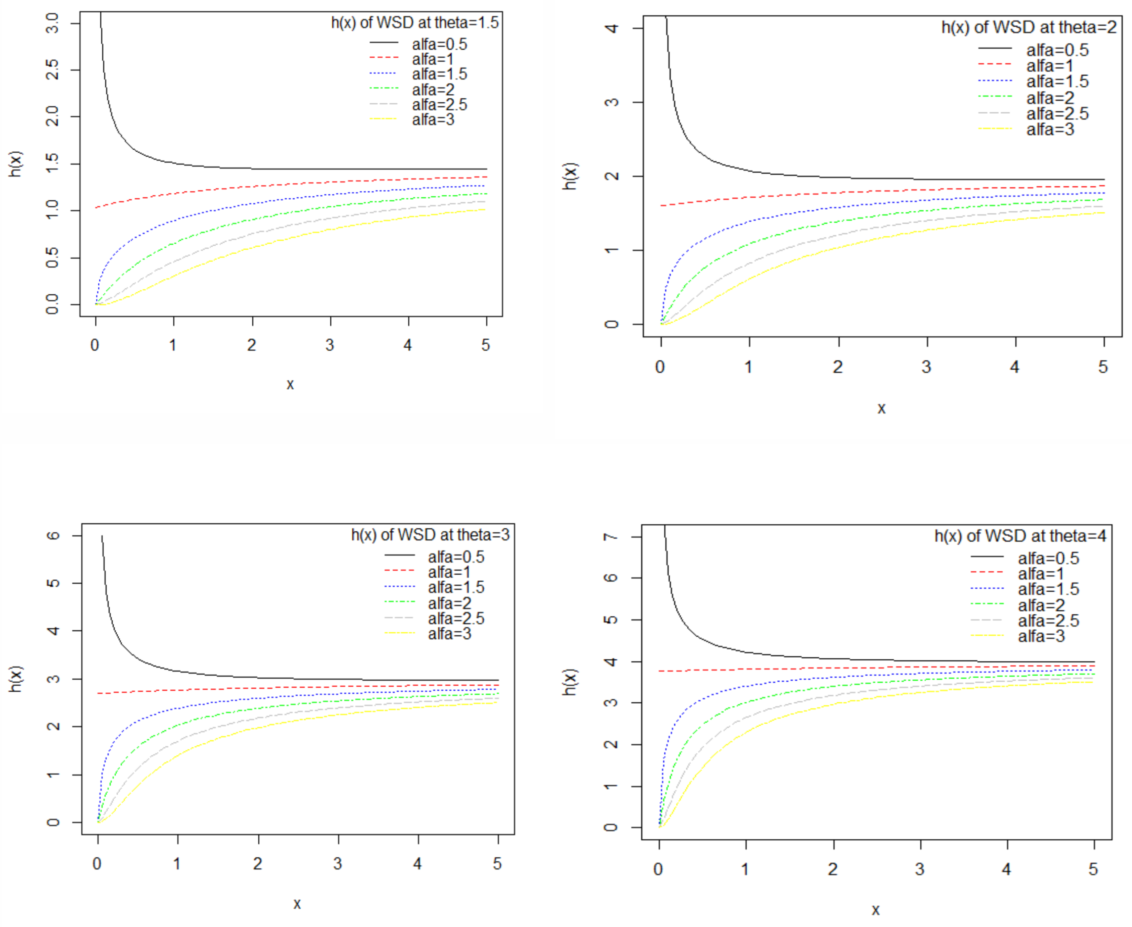


Figure 2. Shapes of the hazard rate function, $h(x)$ of the WSD for varying values of parameter (θ,α)

4.2. Mean Residual Life Function

The mean residual life function $m(x)=E(X-x\,|\,X>x)$, using the mixture representation (2.3), of the WSD can be obtained as

$$m(x)=\frac{1}{S(x)}\int_x^\infty y\,f(y)\,dy-x$$

$$=\frac{1}{S(x)}\left[p\int_x^\infty y\,f_1(y;\theta,\alpha)\,dy+(1-p)\int_x^\infty y\,f_2(y;\theta,\alpha+1)\,dy\right]-x$$

$$=\frac{(\theta^2+\alpha+1)(\theta x)^\alpha e^{-\theta x}+\left[\alpha(\theta^2+\alpha+1)-\theta x(\theta^2+\alpha)\right]\Gamma(\alpha,\theta x)}{\theta\left[(\theta^2+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^\alpha e^{-\theta x}\right]} \qquad (4.2.1)$$

The behavior of $m(x)$ at $x=0$ and $x=\infty$, respectively, are thus given by

$$m(0)=\frac{\alpha(\theta^2+\alpha+1)}{\theta(\theta^2+\alpha)}, \quad m(\infty)=\frac{1}{h(\infty)}=\frac{1}{\theta}$$

The shapes of the mean residual life function, $m(x)$ of the WSD for varying values of parameters (θ, α) are shown in figure 3. From the graphs of $m(x)$, it is obvious that for values of $\alpha \leq 0.5$ and $\theta > 0$, $m(x)$ shows upside-down bathtub feature, and for values of $\alpha > 0.5$ and $\theta > 0$, $m(x)$ is monotonically decreasing. The upside-down bathtub feature of $m(x)$ of WSD is particularly useful for modeling engineering reliability data from burn-in-studies.

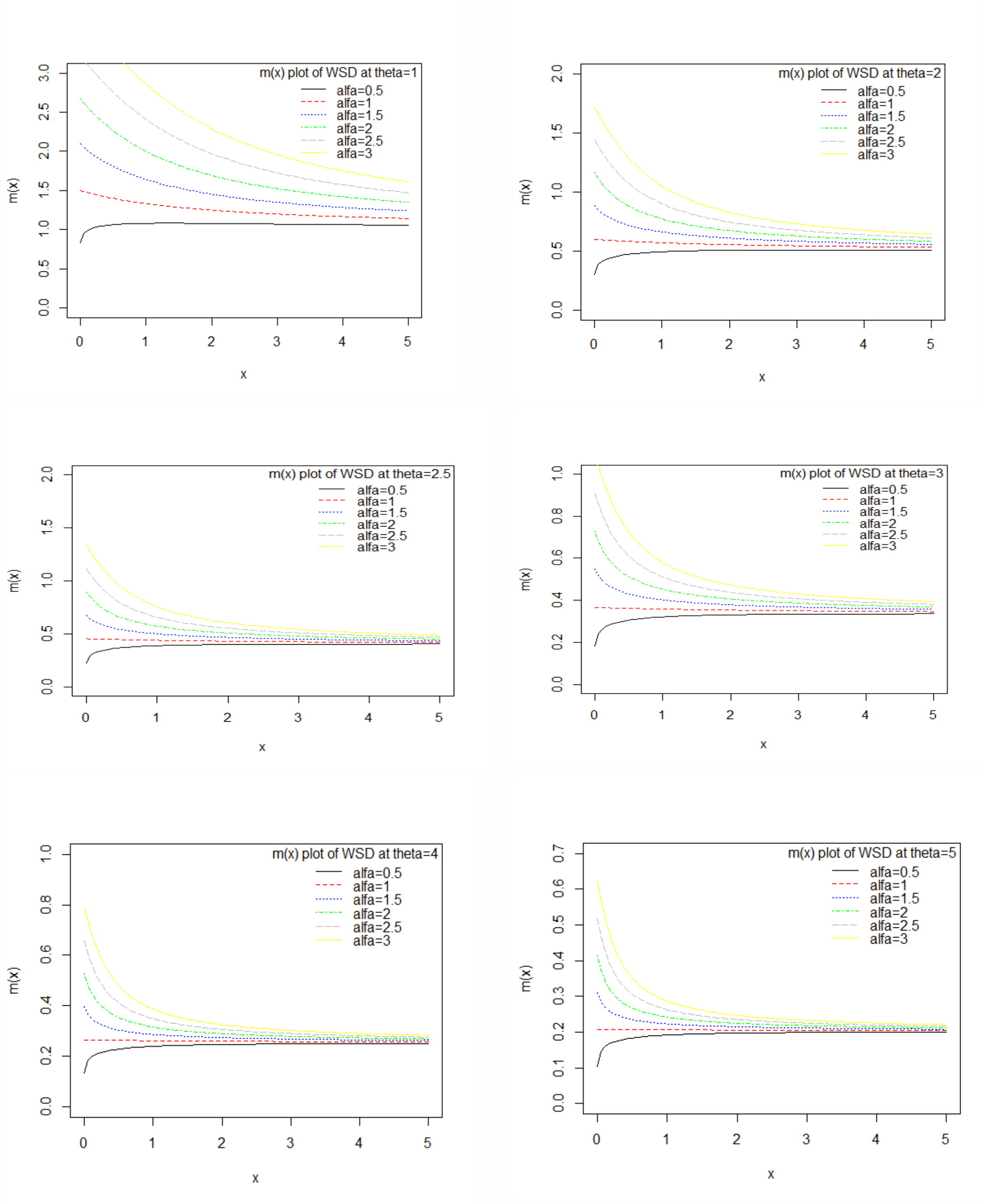


Figure 3. Shapes of the mean residual life function, $m(x)$ of the WSD for varying values of parameters (θ, α)

5. Stochastic ordering

The stochastic ordering of positive continuous random variables is an important tool for judging their comparative behavior. A random variable X is said to be smaller than a random variable Y in the

(i) stochastic order $\left(X \leq_{st} Y\right)$ if $F_X\left(x\right) \geq F_Y\left(x\right)$ for all x

(ii) hazard rate order $\left(X \leq_{hr} Y\right)$ if $h_X\left(x\right) \geq h_Y\left(x\right)$ for all x

(iii) mean residual life order $\left(X \leq_{mrl} Y\right)$ if $m_X\left(x\right) \leq m_Y\left(x\right)$ for all x

(iv) likelihood ratio order $\left(X \leq_{lr} Y\right)$ if $\dfrac{f_X\left(x\right)}{f_Y\left(x\right)}$ decreases in x.

The following interrelationships due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y$$
$$\Downarrow$$
$$X \leq_{st} Y$$

It can be easily shown that WSD is ordered with respect to the strongest 'likelihood ratio' ordering. The stochastic ordering of WSD has been explained in the following theorem 5.1:

***Theorem* 5.1**: Suppose $X \sim$ WSD$\left(\theta_1,\alpha_1\right)$ and $Y \sim$ WSD$\left(\theta_2,\alpha_2\right)$. If $\theta_1 > \theta_2$ and $\alpha_1 = \alpha_2$ (or $\alpha_1 < \alpha_2$ and $\theta_1 = \theta_2$) then $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof: We have

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_1^{\alpha_1-1}\left(\theta_2^{\,2}+\alpha_2\right)\Gamma\left(\alpha_2\right)}{\theta_2^{\alpha_2-1}\left(\theta_1^2+\alpha_1\right)\Gamma\left(\alpha_1\right)} x^{\alpha_1-\alpha_2}\left(\frac{\theta_1+x}{\theta_2+x}\right)e^{-(\theta_1-\theta_2)x} \quad ;\ x>0$$

Now, taking natural logarithm both sides, we get

$$\ln\frac{f_X(x)}{f_Y(x)} = \ln\left[\frac{\theta_1^{\alpha_1-1}\left(\theta_2^{\,2}+\alpha_2\right)\Gamma\left(\alpha_2\right)}{\theta_2^{\alpha_2-1}\left(\theta_1^2+\alpha_1\right)\Gamma\left(\alpha_1\right)}\right]+\left(\alpha_1-\alpha_2\right)\ln x+\log\left(\frac{\theta_1+x}{\theta_2+x}\right)-\left(\theta_1-\theta_2\right)x$$

This gives $$\frac{d}{dx}\ln\frac{f_X(x)}{f_Y(x)} = \frac{\alpha_1-\alpha_2}{x}-\frac{\left(\theta_1-\theta_2\right)}{\left(\theta_1+x\right)\left(\theta_2+x\right)}-\left(\theta_1-\theta_2\right)$$

Thus for $\theta_1 > \theta_2$, and $\alpha_1 = \alpha_2$ (or $\alpha_1 < \alpha_2$ and $\theta_1 = \theta_2$), $\dfrac{d}{dx}\ln\dfrac{f_X(x)}{f_Y(x)} < 0$. This means that $X \leq_{lr} Y$ and hence $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

6. Maximum likelihood estimation of parameters

Suppose $\left(x_1, x_2, x_3, \ldots, x_n\right)$ be a random sample of size n from WSD (2.2). The likelihood function, L of WSD can be obtained as

$$L=\left(\frac{\theta^{\alpha+1}}{\theta^2+\alpha}\right)^n\frac{1}{\left(\Gamma(\alpha)\right)^n}\prod_{i=1}^{n}x_i^{\alpha-1}\left(\theta+x_i\right)e^{-n\theta\bar{x}}$$

The natural log likelihood function is thus obtained as

$$\ln L=n\Big[(\alpha+1)\ln\theta-\ln\left(\theta^2+\alpha\right)-\ln\left(\Gamma(\alpha)\right)\Big]+(\alpha-1)\sum_{i=1}^{n}\ln\left(x_i\right)+\sum_{i=1}^{n}\ln\left(\theta+x_i\right)-n\theta\bar{x}$$

The maximum likelihood estimates (MLE's) $\left(\hat{\theta}\,,\,\hat{\alpha}\right)$ of the parameters $\left(\theta\,,\,\alpha\right)$ of WSD are the solutions of the following non-linear equations

$$\frac{\partial\ln L}{\partial\theta}=\frac{n(\alpha+1)}{\theta}-\frac{2n\theta}{\theta^2+\alpha}+\sum_{i=1}^{n}\frac{1}{\theta+x_i}-n\,\bar{x}=0 \qquad (6.1)$$

$$\frac{\partial\ln L}{\partial\alpha}=n\ln\theta-\frac{n}{\theta^2+\alpha}-n\psi(\alpha)+\sum_{i=1}^{n}\ln\left(x_i\right)=0 \qquad (6.2)$$

where $\psi(\alpha)=\frac{d}{d\alpha}\ln\Gamma(\alpha)$ is the digamma function.

The non-linear equations (6.1) and (6.2) seem to be difficult to solve directly because these maximum likelihood equations are not in closed forms. However, the Fisher's scoring method can be applied to solve these equations. For, we have

$$\frac{\partial^2\ln L}{\partial\theta^2}=-\frac{n(\alpha+1)}{\theta^2}+\frac{2n\left(\alpha-\theta^2\right)}{\left(\theta^2+\alpha\right)^2}-\sum_{i=1}^{n}\frac{1}{\left(\theta+x_i\right)^2}$$

$$\frac{\partial^2\ln L}{\partial\theta\,\partial\alpha}=\frac{n}{\theta}+\frac{2n\theta}{\left(\theta^2+\alpha\right)^2}=\frac{\partial^2\ln L}{\partial\alpha\,\partial\theta}$$

$$\frac{\partial^2\ln L}{\partial\alpha^2}=\frac{n}{\left(\theta^2+\alpha\right)^2}-n\psi'(\alpha)$$

where $\psi'(\alpha)=\frac{d}{d\alpha}\psi(\alpha)$ is the tri-gamma function

For the MLEs $\left(\hat{\theta}\,,\,\hat{\alpha}\right)$ of $\left(\theta\,,\,\alpha\right)$ of WSD (2.2), following equations can be solved

$$\begin{bmatrix}\frac{\partial^2\ln L}{\partial\theta^2} & \frac{\partial^2\ln L}{\partial\theta\,\partial\alpha}\\ \frac{\partial^2\ln L}{\partial\theta\,\partial\alpha} & \frac{\partial^2\ln L}{\partial\alpha^2}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}\begin{bmatrix}\hat{\theta}-\theta_0\\ \hat{\alpha}-\alpha_0\end{bmatrix}=\begin{bmatrix}\frac{\partial\ln L}{\partial\theta}\\ \frac{\partial\ln L}{\partial\alpha}\end{bmatrix}_{\substack{\hat{\theta}=\theta_0\\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α, respectively. These equations are solved iteratively using R-software till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

7. Goodness of fit and applications

In this section, we present the goodness of fit and applications of WSD using maximum likelihood estimates of parameters to two real data sets and compare its fit with the one parameter exponential, Lindley and Shanker distributions and two-parameter WLD. The following two data sets have been considered for the goodness of fit of the proposed distribution.

Data set 1: The following data set represents the waiting times (in minutes) before service of 100 Bank customers and examined and analyzed by Ghitany *et al.,* (2008) for fitting the Lindley (1958) distribution.

0.8	0.8	1.3	1.5	1.8	1.9	1.9	2.1	2.6	2.7	2.9	3.1
3.2	3.3	3.5	3.6	4.0	4.1	4.2	4.2	4.3	4.3	4.4	4.4
4.6	4.7	4.7	4.8	4.9	4.9	5.0	5.3	5.5	5.7	5.7	6.1
6.2	6.2	6.2	6.3	6.7	6.9	7.1	7.1	7.1	7.1	7.4	7.6
7.7	8.0	8.2	8.6	8.6	8.6	8.8	8.8	8.9	8.9	9.5	9.6
9.7	9.8	10.7	10.9	11.0	11.0	11.1	11.2	11.2	11.5	11.9	12.4
12.5	12.9	13.0	13.1	13.3	13.6	13.7	13.9	14.1	15.4	15.4	17.3
17.3	18.1	18.2	18.4	18.9	19.0	19.9	20.6	21.3	21.4	21.9	23.0
27.0	31.6	33.1	38.5								

Data Set 2: The following data represent the tensile strength, measured in GPa, of 69 carbon fibers tested under tension at gauge lengths of 20mm, Bader and Priest (1982)

1.312	1.314	1.479	1.552	1.700	1.803	1.861	1.865	1.944	1.958	1.966	1.997
2.006	2.021	2.027	2.055	2.063	2.098	2.140	2.179	2.224	2.240	2.253	2.270
2.272	2.274	2.301	2.301	2.359	2.382	2.382	2.426	2.434	2.435	2.478	2.490
2.511	2.514	2.535	2.554	2.566	2.570	2.586	2.629	2.633	2.642	2.648	2.684
2.697	2.726	2.770	2.773	2.800	2.809	2.818	2.821	2.848	2.880	2.954	3.012
3.067	3.084	3.090	3.096	3.128	3.233	3.433	3.585	3.585			

Generally the goodness of fit of continuous distributions are based on the values of $-2\ln L$, AIC (Akaike Information Criterion) and K-S Statistic (Kolmogorov-Smirnov Statistic). In order to compare the goodness of fit of the proposed distribution along with other one parameter and two-parameter distributions, values of $-2\ln L$, AIC and K-S of the respective distributions for two real data sets have been computed using maximum likelihood estimates and presented in table 5. The AIC and K-S Statistics are computed using the following formulae:

$AIC = -2\ln L + 2k$ and $\text{K-S} = \sup_x \left| F_n(x) - F_0(x) \right|$, where k = the number of parameters, n = the sample size , $F_n(x)$ is the empirical (sample) cumulative distribution function, and $F_0(x)$ is the theoretical cumulative distribution function. The best distribution is the distribution corresponding to lower values of $-2\ln L$, AIC, and K-S statistics and higher p-value.

Table 5. MLE's, - 2ln L, AIC, K-S Statistic and p-values of the fitted distributions of data sets 1 and 2

	Model	MLEs	$-2\ln L$	AIC	K-S Statistics	p-value
Data 1	WSD	$\hat{\theta}=0.2106$ $\hat{\alpha}=1.1202$	635.04	639.04	0.046	0.984
	WLD	$\hat{\theta}=0.2244$ $\hat{\alpha}=1.3586$	635.59	639.59	0.050	0.964
	Shanker	$\hat{\theta}=0.1983$	635.26	637.26	0.053	0.782
	Lindley	$\hat{\theta}=0.1865$	638.07	640.07	0.068	0.749
	Exponential	$\hat{\theta}=0.1012$	658.04	660.04	0.173	0.005
Data 2	WSD	$\hat{\theta}=9.5997$ $\hat{\alpha}=23.3301$	100.06	104.06	0.058	0.981
	WLD	$\hat{\theta}=9.3756$ $\hat{\alpha}=22.3156$	101.95	105.95	0.059	0.973
	Shanker	$\hat{\theta}=0.6580$	233.01	235.01	0.355	0.005
	Lindley	$\hat{\theta}=0.6590$	238.38	240.38	0.404	0.000
	Exponential	$\hat{\theta}=0.4079$	261.74	263.74	0.448	0.000

It is clear from the close examination of table 5 that WSD is the best model among the one parameter exponential, Lindley and Shanker distributions and two – parameter WLD, since it has the lowest - 2ln L and K-S statistic, and higher p-value.

The variance-covariance matrix and 95% confidence intervals (CI's) for the parameters $\hat{\theta}$ and $\hat{\alpha}$ of WSD for data sets 1 and 2 are presented in table 6.

Table 6. Variance-covariance matrix and 95% confidence intervals (CI's) for the parameters $\hat{\theta}$ and $\hat{\alpha}$

Data set	Parameters	Variance-Covariance Matrix		95% CI	
		$\hat{\theta}$	$\hat{\alpha}$	Lower	Upper
1	$\hat{\theta}$	0.00094	0.007221	0.1561	0.2765
	$\hat{\alpha}$	0.007221	0.070571	0.6517	1.6929
2	$\hat{\theta}$	2.6353	6.3659	6.7676	13.1510
	$\hat{\alpha}$	6.3659	15.7092	16.4018	31.9878

The profile of likelihood estimates for parameters $\hat{\theta}$ and $\hat{\alpha}$ of WSD for two data sets are presented in figures 4(a) and 4(b).

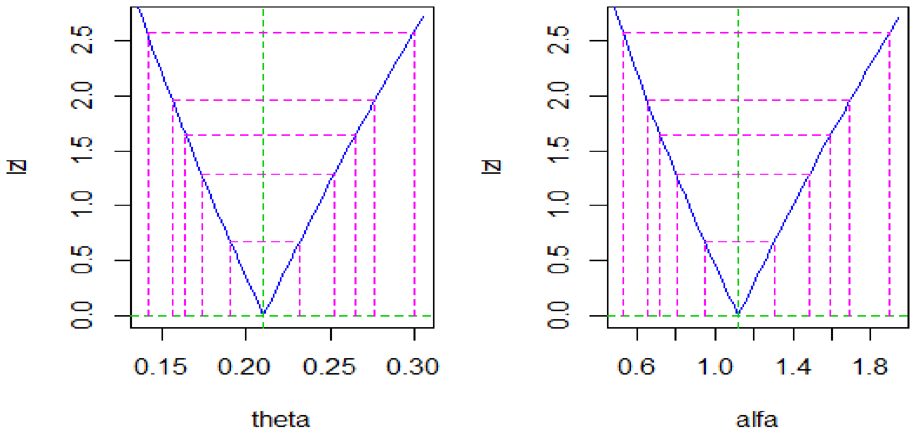


Figure 4(a). Likelihood Estimates for parameters of WSD to data set-1

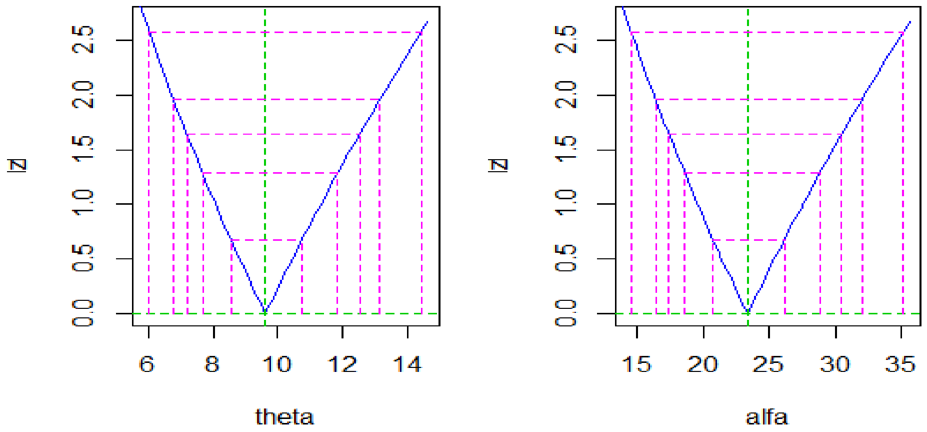


Figure 4(b). Likelihood Estimates for the parameters of WSD to data set-2

The fitted pdf plots for WSD, WLD, Shanker, Lindley and exponential distributions for data sets 1 and 2 are shown in figure 5.

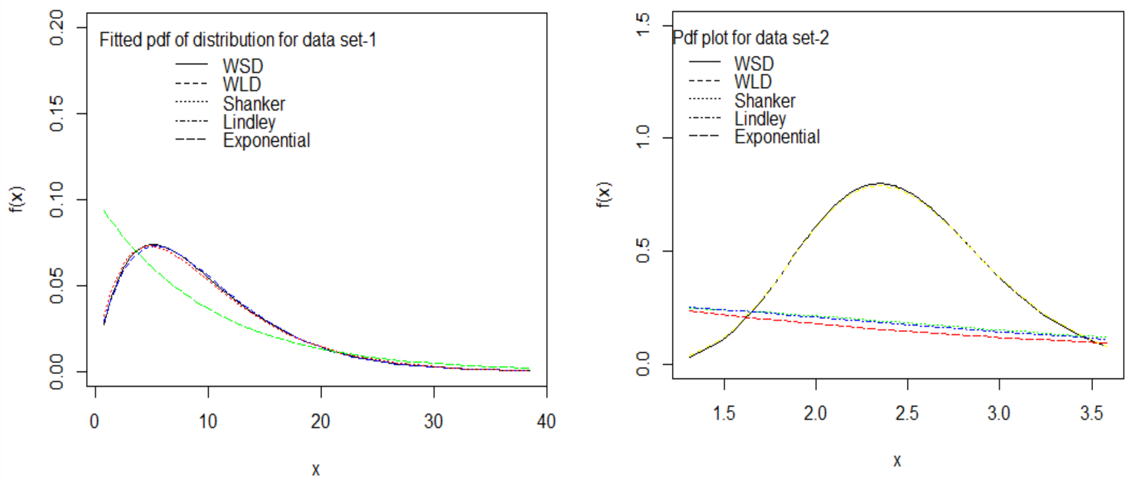


Figure 5. Fitted pdf plots of distributions for data sets 1 and 2

8. Concluding remarks

A two-parameter weighted Shanker distribution (WSD) which includes one parameter Shanker distribution introduced by Shanker (2015) has been proposed for modeling lifetime data. Its structural and Statistical properties including shapes of probability density function for varying values of parameters, moments and moment related measures, hazard rate function, mean residual life function and stochastic ordering have been discussed. The nature of hazard rate function and mean residual life function has been discussed graphically with varying values of parameters. The maximum likelihood estimation for estimating its parameters has been discussed. Variance-covariance matrix and confidence intervals for parameters have been obtained and presented. The goodness of fit of WSD is over one parameter exponential, Lindley and Shanker distributions, and two-parameter WLD has been found to be quite satisfactory.

References

1. Bader, M. G. and Priest, A. M. **Statistical aspects of fiber and bundle strength in hybrid composites,** in Hayashi, T., Kawata, K. and Umekawa, S. (eds.) "Progress in Science in Engineering Composites", ICCM-IV, Tokyo, 1982, pp. 1129–1136.
2. Fisher, R. A. **The effects of methods of ascertainment upon the estimation of frequencies,** Ann. Eugenics, Vol. 6, 1934, pp. 13–25.
3. Ghitany, M. E., Atieh, B. and Nadarajah, S. **Lindley distribution and its Application,** Mathematics Computing and Simulation, Vol. 78, 2008, pp. 493–506.
4. Ghitany, M. E., Alqallaf, F., Al-Mutairi, D. K. and Husain, H. A. **A two-parameter weighted Lindley distribution and its applications to survival data,** Mathematics and Computers in simulation, Vol. 81, 2011, pp.1190-1201.
5. Lindley, D. V. **Fiducial distributions and Bayes' theorem,** Journal of the Royal Statistical Society, Series B, Vol. 20, 1958, pp. 102-107.
6. Patil, G. P. and Rao, C. R. **The Weighted distributions: A survey and their applications,** in Krishnaiah, P. R. "Applications of Statistics", North Holland Publications Co., Amsterdam, 1977, pp. 383-405.
7. Patil, G.P. and Rao, C.R. **Weighted distributions and size-biased sampling with applications to wild-life populations and human families,** Biometrics, Vol. 34, 1978, pp. 179 –189.
8. Rao, C. R. **On discrete distributions arising out of methods of ascertainment**, in Patil, G. P. (ed.) "Classical and Contagious Discrete Distributions", Statistical Publishing Society, Calcutta, 1965, pp. 320–332.
9. Sankaran, M. **The discrete Poisson-Lindley distribution,** Biometrics, Vol. 26, No. 1, 1970, pp. 145–149
10. Shaked, M. and Shanthikumar, J. G. **Stochastic Orders and Their Applications,** Academic Press, New York, 1994.
11. Shanker, R. and Mishra, A. **A quasi Lindley distribution,** African Journal of Mathematics and Computer Science Research, Vol. 6, No. 4, 2013, pp. 64–71.
12. Shanker, R. and Mishra, A. **On Size-Biased Quasi Poisson Lindley Distribution,** International Journal of Probability and Statistics, Vol. 2, No. 2, pp. 28-34.

13. Shanker, R. **Shanker distribution and Its Applications,** International Journal of Statistics and Applications, Vol. 5, No. 6, 2015, pp. 338–348.
14. Shanker, R. **The discrete Poisson-Shanker distribution,** Journal of Biostatistics, Vol. 1, No. 1, 2016, pp. 1-7.
15. Shanker, R. **On Generalized Lindley distribution and its Applications to model lifetime data from biomedical science and engineering,** Insights in Biomedicine, Vol. 1, No. 2, 2016, pp. 1–6.
16. Shanker, R., Hagos, F. and Sujatha, S. **On Modeling of Lifetimes data using Exponential and Lindley distributions,** Biometrics and Biostatistics International Journal, Vol. 2, No. 5, 2015, pp. 1-9.
17. Shanker, R. and Hagos, F. **On Poisson-Lindley distribution and its Applications to Biological Sciences,** Biometrics and Biostatistics International Journal, Vol. 2, No. 4, 2015, pp. 1-5.
18. Shanker, R., Hagos, F. and Sharma, S. **On Quasi Lindley distribution and Its Applications to Model Lifetime Data,** International Journal of Statistical Distributions and Applications, Vol. 2, No. 1, 2016, pp. 1–7.
19. Shanker, R., Hagos, F., Shanker, R., Tekie, A.L., and Simon, S. **On discrete Poisson-Shanker distribution and its applications,** Biometrics and Biostatistics International Journal, Vol. 5, No. 1, 2017, pp. 1-9.
20. Zakerzadeh, H. and Dolati, A. **Generalized Lindley distribution,** Journal of Mathematical extension, Vol. 3, No. 2, 2009, pp. 13–25.