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Contents

THE QUALITY OF INFLUENZA-RELATED ROMANIAN WEBSITES – ARE THEY ANY BETTER THAN SEVEN YEARS AGO?

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Abstract

The proportion of people browsing the Internet for health-related purposes keeps growing and the quality of information users find may have a profound impact on the outcome of their medical decisions. The goal of this study was to observe changes in the quality of the Romanian language influenza-related websites for the general public over a period of seven years. The 2011, baseline sample and the 2018, follow-up sample, included 20 websites each, selected from Google's search results pages using "gripa" (influenza, in Romanian) as a search term. Two independent evaluators followed a common set of detailed instructions and rated the websites for credibility, completeness, and accuracy on a numeric scale going from 0 to 10 points. A number of 16 (80%) of the websites in the baseline sample remained accessible in 2018 but most of them had a major decline in their Google ranking (> 100 positions down). The baseline sample had a mean credibility score of 3.9 points (SD 2.2), a mean completeness scores of 5.8 points (SD 2.8), and a mean accuracy score of 7.5 points (SD 1.2). The follow-up sample had a mean credibility score of 4.1 points (SD 2.6), a mean completeness score of 6.4 points (SD 1.8), and a mean accuracy score of 6.0 points (SD 0.9). Timewise comparison tests detected no change in credibility scores (p>0.05), and completeness scores (p>0.05). Accuracy scores recorded a statistically significant drop (p<0.0001), but considering that the mean difference between the 2011 and 2018 accuracy scores was only 1.5 points, the practical implications of this finding should be interpreted with caution. However, observing these low quality scores and no improvement over such a long period of time, should be a reason for concern for public health professionals. In corroboration with the results of other similar studies, the observed lack of improvement in online health-related information quality should prompt the implementation of interventions aiming to improve the quality of sources used by online health-seekers.

Key words: flu, flu vaccine, health education, health-related information quality, consumer health

Introduction

The Internet has become a major source of information for general public. The proportion of people browsing the Internet for health-related purposes keeps growing (Statista.com, 2014; Fox *et al*., 2013; Andreassen *et al*., 2007). The quality of health information for users may have a profound impact on the outcome of their medical decisions, especially on those related to influenza or other pandemic health problems (Nadaşan, 2016; Covolo *et al*., 2013; Gesualdo *et al*., 2010; Eysenbach *et al*., 2002). The quality of Romanian websites containing influenza-related information was first assessed in 2011 in a cross-sectional study (Nadaşan *et al.*, 2011). The present research is the continuation of the previous one. The main goal of this study was to observe the changes in the quality of the Romanian language influenza-related websites for the general public over a period of seven years (2011-2018). The study also verified if the sites that are compliant with credibility criteria (HON; DISCERN; eEurope, 2002) have higher quality content and if there were any significant changes regarding the quality of influenza main chapters over time.

Material and Methods

The research was designed as an observational longitudinal study. Both, baseline (2011) and follow-up (2018) samples included the first 20 Google Romanian websites containing influenza information for the general public. Google searches were performed using "gripa" (meaning "influenza" in Romanian language) as search-term in 2011 and 2018, respectively. Data acquisition and evaluation were first performed in 2011, and then, at follow-up, in 2018. Two independent assessors evaluated the credibility (HON; DISCERN; eEurope, 2002) of the websites and the completeness and accuracy of the websites' content (Nadaşan, 2018). Credibility, completeness and accuracy decimal scores were calculated for both samples. Mean completeness and accuracy scores were also calculated separately for each chapter of the topic (Definition, Causes and Epidemiology; Symptoms and Complications; Treatment; Prevention). The procedures and calculation methodology are presented in detail in previously published studies (Nadaşan *et al*., 2018; Nadaşan *et al.*, 2016).

Descriptive statistics and mean quality scores were calculated for each sample. The agreement between evaluators was assessed using Cohen's kappa test. Kolmogorov-Smirnov test was used to check the normality of the samples; t-test for independent samples and Mann-Whitney test were used as comparison tests. All statistical analyses were performed in SPSS v. 22. The cut-off value for statistical significance was set at $\alpha = 0.05$.

Results

From the whole sample of 2011 evaluated sites, 16 (80%) were accessible and 4 (20%) were inaccessible in 2018. Moreover, only 6 sites form the 2011 sample remained in the first 100 results of google search in 2018 (2 sites in the first 10 google search results), and only three websites were present in both samples (2011 and 2018).

The mean credibility and quality scores of the websites at baseline and follow-up are presented in Table 1. The compliance with credibility criteria is presented in Table 2. Results of comparing influenza chapters completeness and accuracy mean scores are reported in Table 3.

Table 1. Comparison of mean credibility, completeness and accuracy scores of the two samples

Scores	2011 sample (mean ± SD)	2018 sample (mean ± SD)	p value
Credibility	3.9 ± 2.2	4.1 ± 2.6	0.819
Completeness	5.8 ± 2.8	6.4 ± 1.8	0.6072
Accuracy	7.5 ± 1.2	6.0 ± 0.9	< 0.001

Table 2. Compliance to credibility criteria – comparison between the 2011 and 2018 samples

Credibility criteria	Compliance 2011 (%)	Compliance 2018 (%)	Change (%)
Quality procedure statement	30	44	+14
Displaying date of last update	15	20	+5
Displaying publication date	25	32	+7
Referencing sources	0	36	+36
Authorship disclosure	5	40	+35
Providing contact mechanism	90	44	-46
Confidentiality statement	65	36	-29
Disclosure of commercial interest	20	16	-4
Disclosure of funding	20	0	-20
Consultation disclaimer	50	20	-30
Mission statement	65	28	-37
Owner name and address	85	76	-9

Tabel 3. Quality scores of influenza chapters – comparison between 2011 and 2018 samples

Chapters	Completeness scores			Accuracy scores		
	2011 sample (Mean ± SD)	2018 sample (Mean ± SD)	p value	2011 sample (Mean ± SD)	2018 sample (Mean ± SD)	p value
Definition, Causes and Epidemiology	5.7 ± 3.5	7,1 ± 2.3	0.26	4.5 ± 3.5	4.3 ± 1.9	0.87
Symptoms and complications	6.0 ± 3.4	6.2 ± 2.3	0.79	4.8 ± 3.2	3.9 ± 1.8	0.27
Treatment	5.3 ± 3.0	6.0 ± 2.3	0.42	3.9 ± 2.3	5.9 ± 1.2	<0.01
Prevention	6.1 ± 3.4	6.1 ± 2.1	0.67	5.0 ± 3.4	5.8 ± 1.0	0.79

Discussions

Regarding the evolution of the influenza-related information on Romanian Internet in time, even if 80% were still accessible after 7 years, only two of these were found in the first ten Google results and only three sites are present in both samples (2011 and 2018). The matter of concern is not the almost total modification of Google results in time, but the credibility and quality of the present influenza-related information on the Romanian cyber-space.

Regarding the credibility of the evaluated websites there was no statistical significance between the scores of the two samples. Looking at each credibility criteria, websites from 2018 were found to be more compliant to some particular criteria then those from 2011; the best improvement was found in referencing the sources (36% improvement) and Authorship disclosure (35% improvement). In practical terms, regarding the credibility of the websites, nowadays users search influenza-related information on sites with the same credibility like 7 years ago. Because websites improved some credibility criteria over time, we can say that today the average users can follow the references and authors of the online Romanian information, which can widen a little bit their horizon regarding influenza. As far as the credibility of the sites, similar results were obtained in a 5-year longitudinal study about breast cancer-related information on the Romanian websites (Nadaşan *et al.*, 2018).

Comparing the mean completeness scores of the two samples we found no statistical difference. Statistical significance was found when comparing the accuracy scores of the two samples. The influenza-related information on the Romanian websites declined significantly in terms of accuracy over a period of 7 years.

We also found no statistical significance comparing the mean quality scores on influenza chapters information, except the chapter about treatment ($p<0.01$). With respect to treatment, patients who read about influenza on Romanian websites, seem to be better informed about the vaccines and other adjuvant therapies in 2018 compared to 2011. The practical implication of this isolated improvement should be interpreted with caution since the overall accuracy score has worsened significantly over the same period of time. However, the overall low quality scores coupled with the lack of improvement over such a long period of time should be a mater of concern for the public health professionals.

We searched the Internet for other published researches about online influenza information for users. The published studies used a different methodology than our study and therefore a rigorous comparison is not possible. Nevertheless, one research showed better results than our study, with the conclusion that the majority of the investigated sites gave sufficient information (Covolo *et al*., 2013) while two other studies presented similar results and concluded that relevant information is not easy to find, nor safe on the majority of the websites (Pehora *et al*., 2015, Gesualdo *et al*., 2010). Another Dutch study about the present influenza vaccination information on websites concluded that news media reports tend to be more objective and non-judgmental while social media is more critical with the necessity of flu shot (Lehmann, 2013).

In corroboration with the results of these studies and others on different topics (Nadaşan *et al*., 2016; Bastos *et al*., 2014; Nadaşan *et al*., 2011; Santana *et al*., 2011; Kunst *et al*., 2002; Griffiths *et al*., 2000), the observed lack of improvement in online health-related information quality should prompt the implementation of interventions aiming to improve the quality of sources used by online health-seekers.

Strengths and limitations

To the best of our knowledge this is the first study that evaluated the evolution of influenza-related information for common users on Romanian websites over a 7-year period. The results may help Romanian language health care professionals and users be aware of the lack of quality of online influenza information and the changes appearing on health Internet.

The limitations of the study correspond with those of the ongoing change of the Internet. The replication of the study may lead to different result, because of the continuous modification in Google ranks and websites information. A real limitation of the study is the subjective nature of the evaluators. We tried to minimize the biases using two individual evaluators (medical doctors or medical students) for each website. Another issue that may look like a deficiency of the study is the small number of included sites (20 for each sample), but since most of the Internet users access only the first page of Google results (first 10 results), the large margin of safety we took transformed it in a strength (Granka *et al.*, 2018). As a limitation may also be considered the use of Google as the only search engine in our study, but taking into account that up to 97% of Romanians use Google as their search engine, this decision does not weaken the methodology of this research (Statcounter, 2018). Ultimately, the results cannot be applied to other languages, since the study sample included only Romanian websites.

Conclusions

Overall, the study has shown no improvement in the credibility and completeness of Romanian websites containing influenza-related information, and a decrease in terms of accuracy over a 7-year period.

There were some improvements of the evaluated Romanian websites regarding particular credibility standards, with a significant difference with respect to two important criteria: referencing the sources and authorship disclosure.

As far as the content quality, the treatment section of influenza-related information was the only improved chapter on Romanian websites in a 7-year period of time.

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ANALYZING FACTORIAL EXPERIMENTS WITH A SINGLE COMMON CONTROL GROUP

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Abstract
Researchers generally put a common control group into their experiments in order to determine how effective the treatments are or to compare the effect of their treatments with a baseline. In this study, classical statistical analysis of factorial experiments and a solution way which has been proposed by Winer et al. (1991) have been compared in terms of type I error rate and test power under different experimental conditions. Results of 100,000 simulation study revealed that performing Winer et al. (1991) test is more appropriate in terms of getting reliable results when there is a single common control group in factorial experiments.

Keywords: *Control group; factorial design; type I error; test power; simulation*

1. Introduction

In practice, researchers especially in the field of medicine, agriculture, pharmacy, genetics, social science and some other related sciences commonly put a control group (or baseline) in their experiments in order to investigate if the treatments make a significant affect on interested variable(s) (Kinser and Robins, 2013). The reason is that, there is no other way to see the effect of treatments on the response correctly and reliably. If the researchers do not have a control group, in this case it will not be possible to determine if the treatments have a significant impact. The control group establishes a baseline that the experimental units are compared to and thus, without a control group, researchers will not have anything to compare the experiment's results to. Therefore, since the control group does not receive a treatment, it allows the researchers to eliminate and isolate the effect of the other factors which cannot be able to control or consider (Winer *et al.*, 1991; Pithon, 2013; Bate and Karp, 2014). Considering that the factorial experiments which have a single control group are commonly designed in practice, especially in the fields of agriculture, medicine, biology, aquaculture, forestry etc. It is obvious that statistical analysis of these experiments will be different from that of the analysis of the classical factorial experiments (Winer *et al.*, 1991; Kramer and Font, 2015). However, it is noticed that many researchers have still been performing the classical statistical analysis of the factorial designs even they have a single common control group (Table 2 and 3). Although Winer *et al.* (1991) proposed a different statistical analysis test for such cases, many researchers especially non-statisticians still do not aware of this test. From the light of this point, a comprehensive Monte Carlo Simulation study has been carried out to investigate the performance of the Winer *et al.* (1991) method for analyzing factorial experiments when there is a single common control group. The performance of the Winer *et al.* (1991) test has been compared to the statistical analysis of the classical factorial experiments. An illustrative example has also been put into the manuscript in order to show how data sets of factorial experiments can be analyzed when there is a single common control group.

2. Material and Methods

Pseudo random numbers have been generated from normal (0, 1) and different non-normal distributions (Beta (10, 10), Beta (5,10), Beta (10, 5), and Chi-Sq (3)) for four different types of factorial experiments with 2x2, 3x3, 4x4, and 2x4 under both homogeneity of variances assumption is met and not met. Number of replications in each sub-group have been determined as 3, 4, 5, 10, and 20. Each experimental condition has been simulated 100,000 times. In order to estimate the test power of the Winer *et al.* (1991) test and classical statistical analysis of factorial experiment, two different constant numbers with standard deviation form have been added to the numbers in the control group and the last sub-group in the factorial part of the study. Experimental conditions which have been simulated have been presented in Table 1.

Table 1. Experimental conditions which have been simulated

Distributions	Effect Size	Variance Ratio	Sample Size
Normal (0,1) t (10) Beta (10, 10)	0.0, 0.75, 1.50	1:...:1 1:...:10 1:...:20	3,4,5,10,20

2.1. Statistical Analyses

2.1.1. Classical Analysis of Factorial Designs (CAM)

Experimental design, computational steps of the classical analysis of the factorial design, and degrees of freedom have been presented in Table 2, Table 3, and Table 4 (Winer et al., 1991). Suppose there are two factors namely A and B. If both factors have two levels (a_1, a_2 and b_1, b_2) and there is a single common control group in the experiment, in this case, the experimental design will be as in the Table 2.

Table 2. Experimental design for classical analysis of factorial design

	Control	b_1	b_2
a_1	Y_{001} Y_{002} $\vdots$ Y_{00n}	Y_{111} Y_{112} $\vdots$ Y_{11n}	Y_{121} Y_{122} $\vdots$ Y_{12n}
a_2	Y_{001} Y_{002} $\vdots$ Y_{00n}	Y_{211} Y_{212} $\vdots$ Y_{21n}	Y_{221} Y_{222} $\vdots$ Y_{22n}

Table 3. Computational steps for classical analysis of factorial designs

(1)	$1 = \left(r\sum_{k=1}^{n_0} Y_{00k} + \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n} Y_{ijk}\right)^2/(n_0 + nrc)$ $2 = r\sum_{k=1}^{n_0} Y_{00k}^2 + \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n} Y_{ijk}^2$ $3 = \sum_{i=1}^{r}(\sum_{k=1}^{n_0} Y_{00k} + \sum_{j=1}^{c}\sum_{k=1}^{n} Y_{ijk})^2/(n_0 + nc)$ $4 = (\sum_{k=1}^{n_0} Y_{00k})^2/n_0 + \sum_{j=1}^{c}(\sum_{i=1}^{r}\sum_{k=1}^{n} Y_{ijk})^2/nr$ $5 = r(\sum_{k=1}^{n_0} Y_{00k})^2/n_0 + \sum_{i=1}^{r}\sum_{j=1}^{c}(\sum_{k=1}^{n} Y_{ijk})^2$	
(2)	Source of Variation	Computational formula for SS
	A	$SS_A = (3) - (1)$
	B	$SS_B = (4) - (1)$
	AxB	$SS_{AB} = (5) - (3) - (4) + (1)$
	Within cell	$SS_{Error} = (2) - (5)$
	Total	$SS_{Total} = (2) - (1)$

Note: r and c denote the numbers of row and column, n_0 and n are the number of replications in the control group and each-sub group

Table 4. Source of variation and degrees of freedom for classical analysis of factorial design

Source of variation	df
Between cell	(rc+r)-1
A	r-1
B	(c+1)-1
AxB	(r-1)c
Within cell	rc(n-1)+r(n_0-1)

2.1.2. Analyzing Factorial Experiments When There Is a Single Common Control Group by Using Winer et al (1991) Method (WM)

Experimental design, computational steps of Winer *et al.* (1991) method, and degrees of freedom of that experiment have been given in Table 5, Table 6, and Table 7.

Table 5. Experimental design for Winer *et al*. (1991) method

Control		b_1	b_2
Y_{001} Y_{002} ⋮ Y_{00n}	a_1	Y_{111} Y_{112} ⋮ Y_{11n}	Y_{121} Y_{122} ⋮ Y_{12n}
	a_2	Y_{211} Y_{212} ⋮ Y_{21n}	Y_{221} Y_{222} ⋮ Y_{22n}

Table 6. Computational steps for Winer *et al.* (1991) method

(1)	$1 = \left(\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n} Y_{ijk}\right)^2/nrc$ $2 = \sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n} Y_{ijk}^2$ $3 = \sum_{i=1}^{r}(\sum_{j=1}^{c}\sum_{k=1}^{n} Y_{ijk})^2/nc$ $4 = \sum_{j=1}^{c}(\sum_{i=1}^{r}\sum_{k=1}^{n} Y_{ijk})^2/nr$ $5 = \sum_{i=1}^{r}\sum_{j=1}^{c}(\sum_{k=1}^{n} Y_{ijk})^2/n$	
(2)	Source of Variation	Computational formula for SS
	A	$SS_A = (3) - (1)$
	B	$SS_B = (4) - (1)$
	AxB	$SS_{AB} = SS_{b.cell} - (SS_{cont.vs.\ all} + SS_A + SS_B)$
	Within Cell	$SS_{Error} = (2) - (5) + SS_0$
	Total	$SS_{Total} = (2) - (1)$

Table 7. Source of variation and degrees of freedom Winer *et al.* (1991) method

Source of variation	df
Between cell	(rc+1)-1
Control vs. all others	1
A	r-1
B	c-1
AxB	(r-1)(c-1)
Within cell	rc(n-1)+(n_0-1)

$$SS_0 = \sum_{k=1}^{n_0} Y_{00k}^2 - \frac{\left(\sum_{k=1}^{n_0} Y_{00k}\right)^2}{n_0}$$

$$C = \frac{rcC_0}{n_0} - \frac{\sum_{i=1}^{r}\sum_{j=1}^{c} Y_{ij.}}{n}$$

Where,

C_0: Sum of the observations in the control group

$$SS_{cont.vs.\ all} = \frac{C^2}{[(rc)^2/n_0] + (rc/n)}$$

$$SS_{b.cell} = \frac{C_0^2}{n_0} + \frac{\sum_{i=1}^{r}\sum_{j=1}^{c} Y_{ij.}^2}{n} - \frac{\left(\sum_{i=1}^{r}\sum_{j=1}^{c}\sum_{k=1}^{n} Y_{ijk} + C_0\right)^2}{nrc + n_0}$$

2.2. Illustrative Example.

A data set from an experiment which was carried out in 2007 to investigate the effect of two different feeding programs (R20, NF6) and two lighting programs (16L:8D and 12L:12D) on slaughter weights of Ross 308 broiler chickens was used. There is also a single common control group in this study. The data set has given in Table 8. This data set has been analyzed both by using Classical Analysis Method (CAM) and Winer *et al.* (1991) Method (WM) in order to show differences in the computation steps of two methods.

Table 8. Data set which has been considered for this study

	Feeding Programs		**Control Group**
Lighting Programs	R20	NF6	
16:8	2.216	2.209	
	2.043	1.865	
	2.021	2.452	2.503
	2.311	2.490	2.738
	1.910	1.919	2.701
	1.887	2.215	2.711
12:12	2.484	2.316	2.297
	2.312	1.957	2.085
	2.250	1.894	
	2.201	2.047	
	1.864	1.816	
	2.443	1.836	

2.2.1. Analyzing Data Set by Using Classical Analysis of Factorial Design (CAM)

Experimental Design, computational steps and results of the CAM have been given in Table 9, 10, 11 and 12 respectively.

Table 9. Experimental design for classical analysis of factorial design

		Feeding Programs	
Lighting Programs	Control	R20	NF6
16:8	2.503	2.216	2.209
	2.738	2.043	1.865
	2.701	2.021	2.452
	2.711	2.311	2.490
	2.297	1.910	1.919
	2.085	1.887	2.215
12:12	2.503	2.484	2.316
	2.738	2.312	1.957
	2.701	2.250	1.894
	2.711	2.201	2.047
	2.297	1.864	1.816
	2.085	2.443	1.836

Table 10. Slaughter weight sums of sub-groups

		Feeding Programs		
Lighting Programs	Control	R20	NF6	Σ
16:8	15.035	12.388	13.150	40.573
12:12	15.035	13.554	11.866	40.455
Σ	30.070	25.942	25.016	81.028

Table 11. Computational steps for classical analysis of factorial designs

(1)	$1 = 182.376$ $2 = 185.443$ $3 = 182.376$ $4 = 183.583$ $5 = 183.833$	
(2)	Source of Variation	Computational formula for SS
	Lighting Programs (A)	$SS_A = (3) - (1) = 0.000$
	Feeding Programs (B)	$SS_B = (4) - (1) = 1.207$
	Interaction (AxB)	$SS_{AB} = (5) - (3) - (4) + (1) = 0.250$
	Experimental Error (within cell)	$SS_{Error} = (2) - (5) = 1.610$
	Total	$SS_{Total} = (2) - (1) = 3.067$

Table 12. Results for classical analysis of factorial designs

Source of variation	**df**	**SS**	**MS**	**F**	**P-Value**
Between cell	5	1.457			
Lighting Programs (A)	1	0.000	0.000	0.000	1.000
Feeding Programs (B)	2	1.207	0.604	11.185	**0.000****
Interaction (AxB)	2	0.250	0.125	2.315	0.116
Within cell	30	1.610	0.054		

****P<0.01**

2.2.2. Analyzing Data Set by Using Winer et al (1991) Method (WM)

Experimental Design, computational steps and results of the WM have been given in Table 13, 14, 15 and 16 respectively.

Table 13. Experimental design for Winer *et al.* (1991) method

Control	Lighting Programs	Feeding Programs R20	Feeding Programs NF6
2.503	16:8	2.216	2.209
2.738		2.043	1.865
2.701		2.021	2.452
2.711		2.311	2.490
2.297		1.910	1.919
2.085		1.887	2.215
	12:12	2.484	2.316
		2.312	1.957
		2.250	1.894
		2.201	2.047
		1.864	1.816
		2.443	1.836

Table 14. Sum of sub-groups in terms of slaughter weight

		Feeding Programs		
Control (Σ)	Lighting Programs	R20	NF6	Σ
15.035	16:8	12.388	13.150	25.538
	12:12	13.554	11.866	25.420
	Σ	25.942	25.016	50.958

$$SS_0 = \sum_{k=1}^{n_0} Y_{00k}^2 - \frac{(\sum_{k=1}^{n} Y_{00k})^2}{n_0} = 38.030 - \frac{15.035^2}{6} = 0.355$$

Where , SS_0 is sum of squares for Control group.

Table 15. Computational steps for Winer *et al.* (1991) method

(1)		$1 = 108.197$ $2 = 109.383$ $3 = 108.197$ $4 = 108.232$ $5 = 108.483$
(2)	Source of Variation	Computational formula for SS
	Lighting Programs (A)	$SS_A = (3) - (1) = 0.000$
	Feeding Programs (B)	$SS_B = (4) - (1) = 0.035$
	Interaction (AxB)	$SS_{AB} = SS_{b.cell} - (SS_{cont.vs.all} + SS_A + SS_B) = 0.252$
	Within cell	$SS_{Error} = (2) - (5) + SS_0 = 0.900 + 0.355 = 1.255$

$$C = \frac{rcC_0}{n_0} - \frac{\sum AB_{ij}}{n} = \frac{(2)(2)(15.035)}{6} - \frac{50.958}{6} = 1.530$$

$$SS_{cont.vs.\ all} = \frac{C^2}{[(rc)^2/n_0] + (rc/n)} = \frac{1.530^2}{(4^2/6) + 4/6} = 0.702$$

$$SS_{b.cell} = \frac{C_0^2}{n_0} + \frac{\sum(AB_{ij})^2}{n} - \frac{(G + C_0)^2}{nrc + n_0} = \frac{15.035^2}{6} + 108.483 - \frac{(15.035 + 50.958)^2}{(6)(2)(2) + 6} = 0.989$$

Table 16. Results for Winer *et al.* (1991) method

Source of variation	df	SS	MS	F	*P-Value*
Between cell	4	0.989			
Control vs. all others	1	0.702	0.702	14.040	**0.001****
Lighting Programs (A)	1	0.000	0.000	0.000	1.000
Feeding Programs (B)	1	0.035	0.035	0.700	0.411
Interaction (AxB)	1	0.252	0.252	5.040	**0.003****
Within cell (0.900+0.355)	25	1.255	0.050		

****P<0.01**

3. Results

3.1. Results of Simulation Study

Type I error estimates have been given in Table 17-21, respectively. As it can be seen from Table 17-21, as long as the variances are homogenous, the Winer et al (1991) Method (WM) has given more reliable results in terms of retaining the Type I error rates at the nominal alpha level (0.05) regardless of number of replications, number of factor levels, and distribution shapes. These results are also valid for testing the effect of control group, main and interaction effects. All Type I error estimates have been found very close to 0.05 when the Winer et al (1991) Method is used for analyzing data sets. On the other hand, the preferring the usage of the Classical Statistical Analysis Method (CAM) for analyzing factorial experiments which have a single common control group has led to get much more deviated estimates even when normality and homogeneity of variances assumptions are met. In other word, performing CAM caused to flactuation in Type I error rate and in test power. When the effect of deviations in the homogeneity of the variances on Type I error and test power estimates is examined, it is noticed that non-fulfillment of the homogeneity of variances assumption has caused to not to retain the type I error rates at 5.00%. Both methods have given obviously deviated estimates under these experimental conditions. The Type I error estimates of WM have varied between 6.6 and 17.0% for testing main and interaction effects while the Type I error estimates for testing effect of the control group have been varied between 0.4 and 3.7%. As it is expected, the effect of heterogeneity of variances on the Type I error estimates has been become more obvious especially when the samples are taken from non-normal populations. The effect of the number of factor levels on the Type I error estimates is negligible level as long as the variances are homogeneous.

When both analyses methods (CAM and WM) are compared in terms of test power, it has observed that the test power estimates of both methods have been mainly affected by the number of replications in each sub-group, effect size, number of the factor levels, and whether the variances are homogenous or not. As it is expected, the test power values increased as the number of replications and effect size increased. The test power estimates have not been obviously affected by the deviations from normality as long as the variances are homogeneous. For example, when distributions are normal, both factors have two levels (2x2), variances are homogenous (1:1:..:1), effect size is 0.75, and number of replication is 10, the test power estimates for WM have been found as 34.3%, 21.4 %21.4%, and 21.5% for effect of control group, main effect-A, main effect-B, and interaction effect (AxB) respectively. Under the same conditions, when both factors have three (3x3) and four (4x4) levels, the test power values have been estimated as 51.0%, 15.0%, 15.3%, 19.9% and 57.6%, 11.6%, 11.8%, 17.5% respectively. Under the same conditions when samples are taken from Beta (10, 10), the test power estimates have been found as 34.4%, 21.1%, 21.6%, and 21.3% respectively. The test power values have been estimated as 50.9%, 14.9%, 15.1%, and 19.8% when both factors have three (3x3) levels, and 57.3%, 11.7%, 11.8%, and 17.2% when both factors have four (4x4) levels. When samples are taken from Chi-Square with 3 d.f. distribution, the test power values have been estimates as 34.5%, 22.7%, 22.6%, and 22.6% for 2x2 design, 49.4%, 15.5%, 15.4%, and 20.4% for 3x3 design, and 54.8%, 11.6%, 11.7%, and 17.5% for 4x4 design respectively. As it can be seen from Table 22, 23, 24, 25, and 26, lower test power values have been obtained when variances are not homogeneous and this case has become more obvious when deviation from homogeneity is increased.

3.2. Results of Real Data Set

Results of CAM and WM have been given in Table 12 and Table 16 respectively. When the CAM is used (Table 12), in other way, ignoring of the existence a common control group in the experiment, the main effect of lighting program (P=1.000) and interaction effect (P=0.116) are not found to be statistically significant, whereas the main effect of feeding program is significant (P=0.000). However, as it can be seen from Table 16, when the same data set is analyzed by using WM, the main effects of lighting program (P=1.000) and feeding program (P=0.411) are not found statistically significant whereas the interaction effect is found as statistically significant (P=0.003). As it can be seen from Table 12 and Table 16, different results have been obtained when CAM and WM have been used. Reason for reaching different results is related to considering of existence of a single common control group in the statistical analysis stage or not. Therefore, the use of CAM for analyzing data sets in the experiments where there is a single common control group has caused to mask interaction effect.

4. Discussion

In practice, the researchers generally want to know if the treatment of a particular substance has any effect on the experimental units (animals, patients, plants etc). For such cases, the researchers need something or baseline to compare this effect with. For this purpose, the researchers generally put a control group to their experiments in order to determine if the factors or treatments make a significant impact on the response(s). Although it is practically not possible to completely eliminate effect of all variables on the the results of the experiment, but control group allows the researcher to eliminate variables that can't be controlled in an experiment. Therefore, the control group plays an important role in the experimental process. Especially recently factorial experiments with single or common control are widely designed by the researchers wishing to investigate the effect of two or more factors on interested variable(s). Although factorial experiments with a common control group are commonly designed it is noticed that there is a big problem about the statistical analysis of these kinds of experiments. The problem is the usage of the classical statistical analysis of factorial analysis. In other way, the problem stems from the fact that the statistical analysis is performed by conventional methods (Table 3). Preffering this analysis method is not correct and thus it would not be possible to get reliable results. It is because, performing this analysis methods will cause not to retain Type I error rate at the nominal alpha level. It will also cause to negative changes in test power.

Results of this simulation study revealed that doing statistical analysis by ignoring the fact that there is a common control group in the experiment has caused obtaining unreliable results. Type I error rate could not be retained under any conditions even both homogeneity of variances and normality assumptions were met and number of replications were large (n=20). Test power estimates affected negatively when classical computational steps of statistical analysis were performed. Therefore, the usage of the same sub-group at the each level of the row factor (as if the control group is considered as a level of the column factor) (as in Table 9) has caused to obtain significantly higher Type I error rates than 0.05 for the column factor. On the other hand, it has been caused to obtain significantly lower than 0.05 type I error rates for the row factor and interaction. The test power estimates have also been negatively affected by using CAM.

5. Conclusion

Results of Monte Carlo Simulation Study showed that the usage of the Winer *et al.* (1991) method (WM) in analyzing data sets of factorial experiments when there was a single common control group enabled the researcher to get more correct and reliable results as long as variances are homogeneous. At the same time, since the WM enables the researchers to compare all sub-groups to the control group like multiple comparison procedure, it will be possible to get more detailed and reliable results in terms of effects of interested factors. As a results, it is possible to strongly suggested to authors and researchers to use Winer et al (1991) Method for analyzing their data sets if they have a common single control group in their experimental design.

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Apendixes

Table 17. Type I error rate estimates when samples are taken from normal distributions

		$\sigma_{11}^2:\sigma_{12}^2 \ldots :\sigma_{ij}^2$																				
		1:1:...:1							1:1:...:10							1:1:...:20						
rxc	n	A	B	AxB	**Cont.**	**A**	**B**	**AxB**	A	B	AxB	**Cont.**	**A**	**B**	**AxB**	A	B	AxB	**Cont.**	**A**	**B**	**AxB**
2x2	3	2.5	10.3	2.7	**4.9**	**5.0**	**5.1**	**5.1**	6.2	11.2	7.8	**2.6**	**9.5**	**9.5**	**9.5**	8.8	12.9	10.9	**2.5**	**12.2**	**11.9**	**12.0**
	4	2.2	10.5	2.5	**5.0**	**5.0**	**5.1**	**5.1**	5.7	10.6	7.4	**2.1**	**9.1**	**9.1**	**9.1**	7.6	11.9	10.0	**1.8**	**10.9**	**10.9**	**11.1**
	5	2.1	10.7	2.2	**5.1**	**5.1**	**5.1**	**4.9**	5.2	10.0	6.7	**1.7**	**8.6**	**8.7**	**8.4**	7.1	11.3	9.5	**1.4**	**10.5**	**10.5**	**10.4**
	10	1.8	10.7	2.0	**5.1**	**4.9**	**5.0**	**4.9**	4.4	9.1	6.1	**1.2**	**7.8**	**7.7**	**7.8**	5.5	9.5	8.0	**0.7**	**8.9**	**8.9**	**8.8**
	20	1.8	10.8	1.9	**4.9**	**5.1**	**5.2**	**5.0**	3.9	8.5	5.6	**0.9**	**7.4**	**7.4**	**7.3**	4.8	8.7	7.2	**0.4**	**8.1**	**8.2**	**8.0**
3x3	3	2.7	15.4	2.8	**5.1**	**5.0**	**5.0**	**5.0**	6.1	14.4	8.8	**2.3**	**8.9**	**8.9**	**11.0**	9.2	15.7	13.2	**1.6**	**12.0**	**12.1**	**14.8**
	4	2.4	15.4	2.4	**4.9**	**5.0**	**5.0**	**5.0**	5.4	13.7	8.3	**1.9**	**8.4**	**8.4**	**10.6**	8.1	14.2	12.6	**1.2**	**11.1**	**10.8**	**14.2**
	5	2.3	15.5	2.3	**4.9**	**5.1**	**5.0**	**5.1**	5.1	13.2	8.1	**1.7**	**8.2**	**8.1**	**10.3**	7.7	13.6	12.0	**1.0**	**10.5**	**10.4**	**13.5**
	10	2.0	15.7	2.0	**5.1**	**4.9**	**4.9**	**5.0**	4.7	12.4	7.7	**1.3**	**7.5**	**7.3**	**9.8**	6.4	12.0	10.6	**0.6**	**9.1**	**9.1**	**12.0**
	20	1.9	15.6	1.9	**5.0**	**5.0**	**4.9**	**5.2**	4.3	11.9	7.2	**1.2**	**7.2**	**7.1**	**9.3**	5.9	11.1	10.3	**0.4**	**8.4**	**8.4**	**11.5**
4x4	3	2.7	19.4	2.7	**5.1**	**4.9**	**5.0**	**4.9**	5.4	17.5	8.6	**2.5**	**8.0**	**8.0**	**11.2**	8.5	18.1	13.7	**1.8**	**11.2**	**11.2**	**16.0**
	4	2.5	19.5	2.5	**4.9**	**5.0**	**5.0**	**5.0**	5.0	17.5	8.2	**2.3**	**7.7**	**7.7**	**11.0**	7.7	17.2	13.2	**1.4**	**10.3**	**10.5**	**15.5**
	5	2.4	19.7	2.1	**4.9**	**5.0**	**5.0**	**5.0**	4.8	17.1	7.9	**2.2**	**7.4**	**7.4**	**10.7**	7.1	16.5	12.5	**1.3**	**9.8**	**9.7**	**14.8**
	10	2.2	19.8	2.0	**5.1**	**5.0**	**5.0**	**5.0**	4.4	16.5	7.6	**1.9**	**7.0**	**7.0**	**10.3**	6.5	15.0	11.7	**0.9**	**8.9**	**8.8**	**13.7**
	20	2.1	20.2	1.8	**5.1**	**5.0**	**4.9**	**5.1**	4.2	16.2	7.2	**1.9**	**6.8**	**6.6**	**9.9**	6.0	14.5	11.2	**0.7**	**8.2**	**8.2**	**13.1**
2x4	3	3.2	9.4	3.2	**5.0**	**5.0**	**5.0**	**4.9**	5.8	12.3	8.8	**2.3**	**7.8**	**10.7**	**10.7**	7.9	14.9	12.6	**1.7**	**10.0**	**14.2**	**14.1**
	4	3.0	9.6	3.0	**5.0**	**5.1**	**4.9**	**5.0**	5.3	11.4	8.2	**1.9**	**7.2**	**9.9**	**10.0**	7.0	13.8	11.7	**1.2**	**9.0**	**13.2**	**13.1**
	5	3.0	9.6	2.9	**5.0**	**4.9**	**5.0**	**5.0**	5.0	11.0	7.9	**1.6**	**7.0**	**9.6**	**9.7**	6.4	13.1	11.1	**1.0**	**8.4**	**12.5**	**12.5**
	10	2.9	9.7	2.7	**5.0**	**4.9**	**5.0**	**5.0**	4.3	10.2	7.3	**1.3**	**6.4**	**9.0**	**9.0**	5.3	11.5	9.8	**0.5**	**7.1**	**11.2**	**11.2**
	20	2.9	10.0	2.6	**5.2**	**5.0**	**5.0**	**4.9**	4.2	9.6	6.9	**1.1**	**6.1**	**8.5**	**8.5**	4.7	11.0	9.3	**0.4**	**6.6**	**10.7**	**10.6**

Bold: Results of WM
Regular: Results of CAM

Table 18. Type I error rate estimates when samples are taken from Beta(10,10) distributions

		$\sigma_{11}^2:\sigma_{12}^2\dots:\sigma_{ij}^2$																				
		1:1:...:1							1:1:...:10							1:1:...:20						
rxc	n	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB
2x2	3	2.5	10.5	2.7	4.9	5.1	5.1	5.0	6.6	11.6	8.2	2.6	9.9	9.9	10.0	9.1	13.3	11.3	2.5	12.4	12.5	12.5
	4	2.2	10.7	2.4	5.0	5.0	5.1	5.1	5.9	10.8	7.5	2.1	9.4	9.4	9.3	8.0	12.3	10.4	1.9	11.4	11.4	11.5
	5	2.2	10.7	2.3	5.0	5.2	5.1	5.0	5.4	10.2	7.1	1.8	8.8	8.9	8.8	7.1	11.2	9.5	1.4	10.4	10.4	10.6
	10	1.9	10.6	2.0	5.0	5.0	4.9	4.9	4.4	9.0	6.2	1.1	7.9	7.9	7.9	5.4	9.4	7.9	0.7	8.8	8.8	8.8
	20	1.7	10.7	1.9	5.0	5.0	5.0	4.9	4.0	8.4	5.6	0.9	7.3	7.3	7.2	4.8	8.7	7.1	0.4	8.0	8.1	8.0
3x3	3	2.7	15.4	2.8	5.0	5.1	5.0	5.0	6.2	14.6	9.1	2.2	9.1	9.1	11.2	9.5	16.1	13.7	1.7	12.3	12.4	15.4
	4	2.4	15.8	2.5	5.0	5.1	5.2	5.1	5.7	13.9	8.7	1.9	8.6	8.6	10.9	8.3	14.6	12.6	1.3	11.1	11.2	14.2
	5	2.3	15.7	2.2	5.0	5.1	5.0	5.0	5.4	13.2	8.1	1.7	8.3	8.2	10.4	7.6	13.6	11.9	1.0	10.5	10.5	13.5
	10	2.1	15.8	2.0	5.0	5.2	5.0	4.9	4.7	12.5	7.7	1.4	7.6	7.6	9.8	6.5	11.9	10.8	0.6	9.1	9.0	12.1
	20	1.9	15.9	1.8	5.1	5.0	5.1	5.1	4.2	12.0	7.1	1.2	7.1	7.1	9.2	5.8	11.1	10.2	0.4	8.4	8.4	11.4
4x4	3	2.7	19.8	2.8	5.0	5.0	5.0	5.2	5.6	17.6	8.9	2.4	8.2	8.2	11.5	8.8	18.3	14.0	1.7	11.6	11.5	16.3
	4	2.5	19.7	2.3	5.0	5.0	4.9	5.0	5.2	17.4	8.4	2.2	7.8	7.9	11.2	8.1	17.1	13.2	1.4	10.8	10.4	15.4
	5	2.4	20.0	2.3	5.0	5.0	5.1	5.2	4.7	17.3	8.0	2.1	7.5	7.7	10.9	7.5	16.7	12.7	1.2	10.0	10.0	14.8
	10	2.2	19.6	1.8	4.8	4.9	5.0	4.9	4.3	16.5	7.7	1.9	7.0	7.1	10.4	6.6	15.2	11.9	0.9	9.0	9.0	13.8
	20	2.1	19.8	1.7	4.8	5.0	4.9	5.1	4.2	16.2	7.4	1.8	6.7	6.6	10.0	6.0	14.5	11.2	0.7	8.3	8.3	13.1
2x4	3	3.3	9.4	3.2	4.8	5.0	5.1	5.1	6.0	12.4	9.0	2.2	8.0	10.9	10.8	8.2	15.2	13.0	1.7	10.3	14.5	14.5
	4	3.1	9.5	3.1	4.9	5.0	5.1	5.1	5.3	11.6	8.6	1.8	7.3	10.2	10.5	7.0	14.0	11.9	1.2	9.1	13.3	13.4
	5	3.0	9.7	2.9	5.0	5.1	5.1	5.0	5.0	11.3	8.2	1.6	7.0	9.9	9.9	6.5	13.4	11.3	1.0	8.6	12.8	12.7
	10	3.0	9.8	2.8	5.0	5.1	5.0	5.0	4.4	10.1	7.5	1.2	6.4	8.9	9.3	5.3	11.6	10.0	0.5	7.2	11.2	11.4
	20	2.9	9.7	2.6	4.9	5.1	5.0	4.9	4.2	9.7	6.9	1.1	6.1	8.6	8.6	4.7	10.9	9.3	0.4	6.6	10.5	10.6

Table 19. Type I error rate estimates when samples are taken from Beta(10,5) distributions

		$\sigma_{11}^2:\sigma_{12}^2\dots:\sigma_{ij}^2$																				
		1:1:...:1							1:1:...:10							1:1:...:20						
rxc	n	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB
2x2	3	2.5	10.7	2.7	5.1	5.1	5.0	5.1	6.7	11.7	8.4	2.6	10.2	10.1	10.1	9.3	13.8	11.6	2.7	12.7	12.7	12.8
	4	2.2	10.7	2.4	4.9	4.9	5.1	5.0	6.0	10.8	7.7	2.2	9.5	9.4	9.4	8.0	12.3	10.4	1.9	11.4	11.4	11.4
	5	2.1	10.5	2.3	4.9	5.0	5.0	5.0	5.5	10.3	7.3	1.9	9.0	8.9	9.1	7.3	11.4	9.5	1.5	10.7	10.7	10.5
	10	1.9	10.6	2.0	5.0	5.1	5.0	5.0	4.6	9.2	6.3	1.3	8.1	8.0	8.0	5.7	9.6	8.1	0.8	9.0	8.9	9.1
	20	1.8	10.6	1.9	4.9	5.0	4.9	5.0	4.1	8.6	5.9	0.9	7.5	7.5	7.5	5.0	8.7	7.3	0.5	8.3	8.1	8.1
3x3	3	2.6	15.6	2.9	5.0	5.0	5.1	5.1	6.3	14.4	9.3	2.3	9.3	9.2	11.4	9.6	16.1	14.0	1.7	12.6	12.6	15.8
	4	2.4	15.6	2.5	5.0	5.0	5.0	5.1	5.7	13.8	8.5	1.8	8.6	8.7	10.8	8.4	14.6	12.9	1.3	11.3	11.3	14.4
	5	2.3	15.6	2.3	5.0	5.1	5.0	5.0	5.4	13.6	8.3	1.7	8.3	8.3	10.5	7.8	14.0	12.3	1.1	10.7	10.8	13.9
	10	2.1	15.7	2.0	5.0	5.0	5.0	4.9	4.6	12.5	7.6	1.3	7.5	7.7	9.7	6.4	11.8	10.9	0.6	9.0	9.0	12.4
	20	2.0	15.9	1.8	5.0	4.9	5.1	4.9	4.3	12.2	7.3	1.2	7.1	7.2	9.4	6.0	11.4	10.1	0.4	8.4	8.6	11.5
4x4	3	2.8	19.7	2.9	5.0	5.1	5.1	5.2	5.6	17.7	8.9	2.5	8.2	8.3	11.6	9.1	18.6	14.4	1.7	11.8	11.7	16.6
	4	2.5	19.7	2.4	4.8	5.0	5.0	5.0	5.2	17.4	8.4	2.2	7.9	7.8	11.2	8.1	17.3	13.6	1.4	10.8	10.8	15.8
	5	2.4	19.7	2.2	4.9	5.0	5.0	5.0	5.0	17.1	8.2	2.1	7.7	7.6	10.9	7.5	16.7	13.0	1.2	10.2	10.2	15.2
	10	2.3	19.9	1.8	5.0	5.1	5.0	5.0	4.4	16.6	7.7	1.9	7.1	7.1	10.5	6.7	15.4	12.0	0.9	9.1	8.9	14.0
	20	2.2	20.0	1.7	5.0	5.1	5.1	5.0	4.2	16.4	7.4	1.8	6.8	6.8	10.1	6.1	14.4	11.3	0.7	8.4	8.3	13.2
2x4	3	3.2	9.5	3.2	4.9	5.0	5.0	5.0	6.0	12.3	9.1	2.2	8.0	10.8	10.9	8.2	15.5	13.3	1.7	10.4	14.8	14.9
	4	3.1	9.6	3.1	5.0	4.9	5.0	5.1	5.3	11.8	8.6	1.9	7.4	10.4	10.4	7.3	14.2	12.1	1.2	9.3	13.6	13.6
	5	3.1	9.5	2.9	5.0	5.0	4.9	5.0	5.0	11.3	8.1	1.6	7.0	10.0	9.9	6.6	13.6	11.5	1.0	8.6	13.0	12.9
	10	3.0	9.6	2.8	4.9	5.1	4.9	5.1	4.4	10.3	7.4	1.3	6.3	9.1	9.1	5.6	11.7	10.1	0.5	7.4	11.4	11.4
	20	2.9	9.8	2.7	5.1	5.1	5.0	5.0	4.2	9.8	7.0	1.1	6.1	8.7	8.7	4.7	10.8	9.4	0.4	6.6	10.5	10.7

Table 20. Type I error rate estimates when samples are taken from Beta(5,10) distributions

		$\sigma_{11}^2:\sigma_{12}^2\dots:\sigma_{ij}^2$																				
		1:1:...:1							1:1:...:10							1:1:...:20						
rxc	n	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB
2x2	3	2.5	10.5	2.7	4.9	5.1	5.1	5.0	6.7	11.9	8.4	2.6	10.1	10.3	10.1	9.3	13.7	11.7	2.6	12.8	12.8	12.8
	4	2.3	10.6	2.4	5.0	5.1	4.9	5.0	5.9	10.9	7.7	2.2	9.3	9.4	9.4	8.2	12.3	10.5	2.0	11.7	11.4	11.6
	5	2.1	10.8	2.3	5.1	5.0	5.1	5.0	5.4	10.3	7.2	1.8	8.9	8.9	8.9	7.2	11.4	9.6	1.5	10.5	10.6	10.6
	10	1.9	10.6	2.0	5.0	5.0	4.8	5.0	4.3	9.1	6.1	1.2	7.8	7.7	7.9	5.6	9.6	8.0	0.7	8.9	8.9	9.0
	20	1.6	10.6	1.9	4.8	4.9	5.1	4.9	3.9	8.5	5.8	0.9	7.3	7.5	7.3	4.9	8.7	7.3	0.5	8.2	8.2	8.2
3x3	3	2.6	15.4	2.7	4.9	5.0	5.0	5.0	6.4	14.4	9.3	2.3	9.3	9.2	11.6	9.5	16.2	13.7	1.8	12.5	12.5	15.4
	4	2.3	15.5	2.5	4.9	4.9	5.0	5.1	5.6	13.7	8.7	1.9	8.5	8.7	10.9	8.6	14.5	12.8	1.3	11.5	11.3	14.4
	5	2.2	15.5	2.3	4.8	4.9	5.0	4.9	5.4	13.6	8.4	1.8	8.3	8.4	10.7	8.0	13.8	12.2	1.0	10.8	10.6	13.8

	10	2.0	15.9	1.9	4.9	4.9	5.0	4.9	4.7	12.4	7.6	1.3	7.6	7.5	9.7	6.7	12.2	10.9	0.6	9.2	9.4	12.2
	20	2.0	15.9	1.9	5.1	5.0	4.9	5.2	4.3	11.9	7.2	1.2	6.9	7.0	9.3	6.0	11.3	10.3	0.4	8.6	8.5	11.6
4x4	3	2.8	19.8	2.8	4.9	5.1	5.1	5.2	5.7	17.7	8.9	2.5	8.3	8.3	11.6	8.8	18.3	14.4	1.7	11.5	11.5	16.6
	4	2.6	19.8	2.5	5.0	5.0	5.0	5.1	5.1	17.2	8.5	2.3	7.9	7.9	11.3	8.0	17.4	13.3	1.4	10.7	10.7	15.6
	5	2.4	19.9	2.3	5.0	5.0	5.0	5.0	4.9	17.1	8.3	2.1	7.5	7.6	11.0	7.5	16.6	12.9	1.2	10.2	10.1	15.0
	10	2.2	20.2	1.9	5.0	5.0	5.0	5.1	4.5	16.6	7.7	2.0	7.1	7.1	10.5	6.5	15.3	11.9	0.9	8.9	9.0	13.9
	20	2.2	20.1	1.7	5.1	5.1	5.0	4.9	4.1	16.2	7.3	1.8	6.6	6.7	10.0	6.1	14.5	11.3	0.8	8.4	8.3	13.2
2x4	3	3.2	9.5	3.2	5.1	5.1	5.0	5.0	5.8	12.5	9.1	2.2	7.8	10.9	10.9	8.3	15.5	13.2	1.8	10.4	14.8	14.8
	4	3.1	9.7	3.0	5.0	5.0	5.0	5.0	5.4	11.6	8.5	1.9	7.4	10.2	10.3	7.2	14.4	12.3	1.3	9.3	13.8	13.7
	5	3.1	9.8	2.9	4.9	5.1	5.2	5.0	5.2	11.5	8.4	1.7	7.2	10.2	10.1	6.7	13.4	11.3	1.0	8.7	12.8	12.7
	10	2.9	9.7	2.8	5.0	5.0	4.9	5.0	4.4	10.2	7.5	1.2	6.4	9.1	9.2	5.4	11.7	10.0	0.5	7.3	11.4	11.3
	20	2.9	9.7	2.6	5.0	5.0	4.9	5.0	4.1	9.8	7.0	1.1	6.0	8.7	8.7	4.8	11.1	9.4	0.4	6.7	10.8	10.7

Table 21. Type I error rate estimates when samples are taken from Chi-Sq(3) distributions

		$\sigma_{11}^2:\sigma_{12}^2 \dots :\sigma_{ij}^2$																				
		1:1:…:1							1:1:…:10							1:1:…:20						
rxc	n	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB
2x2	3	2.3	9.2	2.7	4.4	4.6	4.7	4.6	7.2	12.3	8.9	3.3	10.9	10.9	10.8	10.9	15.6	13.3	3.7	14.7	14.7	14.6
	4	2.1	9.3	2.3	4.5	4.6	4.6	4.6	6.5	11.6	8.4	2.9	10.4	10.3	10.4	9.8	14.4	12.4	3.1	13.6	13.5	13.6
	5	2.1	9.7	2.2	4.5	4.7	4.7	4.6	6.1	11.3	8.0	2.5	9.8	10.0	9.8	8.8	13.5	11.6	2.7	12.6	12.6	12.7
	10	1.8	10.5	2.1	4.9	4.9	4.8	4.9	5.1	10.0	6.8	2.0	8.7	8.6	8.6	6.9	11.4	9.4	1.7	10.5	10.6	10.5
	20	1.8	10.4	2.0	4.9	5.0	4.9	5.0	4.4	9.1	6.3	1.4	8.0	8.0	7.9	5.7	9.9	8.2	1.0	9.1	9.2	9.1
3x3	3	2.5	13.6	3.1	4.5	4.6	4.6	4.8	6.0	13.8	8.9	2.9	9.0	9.0	11.1	10.1	16.6	14.9	2.5	13.3	13.4	16.8
	4	2.3	14.3	2.7	4.5	4.6	4.7	4.7	5.7	13.4	8.7	2.6	8.7	8.6	11.1	9.0	15.5	13.8	2.0	12.2	12.4	15.8
	5	2.3	14.5	2.4	4.5	4.8	4.8	4.7	5.5	13.4	8.5	2.3	8.5	8.5	10.9	8.6	15.1	13.4	1.9	11.9	11.9	15.2
	10	2.1	15.5	2.2	4.7	4.9	5.0	4.9	4.8	13.0	7.8	1.9	7.7	8.0	10.1	7.2	13.1	11.9	1.2	10.1	10.2	13.5
	20	2.0	15.6	2.0	4.8	5.0	5.0	4.9	4.5	12.3	7.6	1.5	7.4	7.4	9.7	6.5	12.1	11.1	0.8	9.2	9.2	12.5
4x4	3	2.8	17.8	3.1	4.6	4.7	4.7	4.8	5.3	16.4	8.1	3.1	7.8	7.7	10.6	8.6	18.2	14.4	2.5	11.4	11.6	17.0
	4	2.6	18.4	2.7	4.4	4.7	4.7	4.8	5.0	16.5	8.2	2.8	7.6	7.7	10.9	8.0	17.5	13.8	2.3	10.8	10.9	16.3
	5	2.5	19.0	2.5	4.5	4.7	4.8	4.8	4.8	16.5	8.0	2.7	7.4	7.4	10.9	7.6	16.9	13.5	1.9	10.4	10.4	15.9
	10	2.3	19.7	2.2	4.7	4.8	4.8	5.0	4.5	16.7	7.8	2.3	7.2	7.2	10.7	7.1	15.8	12.7	1.5	9.6	9.6	14.9
	20	2.2	20.1	1.9	4.8	5.0	5.0	4.9	4.1	16.2	7.4	2.1	6.7	6.8	10.2	6.3	15.0	11.9	1.1	8.8	8.8	14.0
2x4	3	3.1	8.2	3.3	4.5	4.8	4.6	4.7	5.9	12.2	8.8	2.9	7.9	10.9	10.8	8.7	16.6	14.1	2.5	11.1	15.8	15.8
	4	3.0	8.5	3.0	4.6	4.8	4.7	4.7	5.4	12.1	8.7	2.6	7.5	10.7	10.7	8.0	15.8	13.3	2.2	10.2	15.1	15.0
	5	3.0	8.7	2.8	4.4	4.8	4.6	4.5	5.1	11.8	8.3	2.4	7.2	10.3	10.3	7.4	15.2	12.9	1.8	9.6	14.4	14.5
	10	3.1	9.3	2.6	4.7	5.1	4.9	4.7	4.7	10.9	7.8	1.7	6.7	9.9	9.6	6.1	13.0	11.1	1.3	8.2	12.6	12.5
	20	2.9	9.5	2.7	4.8	5.0	4.9	4.9	4.4	10.2	7.4	1.5	6.3	9.1	9.1	5.3	11.8	10.1	0.7	7.1	11.4	11.4

Table 22. Test power estimates when samples are taken from Normal distributions

			$\sigma_{11}^2:\sigma_{12}^2 \dots :\sigma_{ij}^2$																				
			1:1:…:1							1:1:…:10							1:1:…:20						
rxc	δ	n	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB
2x2	0.75	3	4.8	22.2	5.5	12.3	9.0	9.0	9.1	7.8	16.3	9.6	5.2	11.5	11.5	11.4	9.7	16.3	11.8	4.4	13.1	13.2	13.0
		4	5.6	27.2	6.4	15.7	10.7	10.8	10.8	7.3	17.4	9.5	5.5	11.3	11.4	11.3	8.7	15.9	11.1	3.9	12.3	12.3	12.2
		5	6.3	32.2	7.6	18.8	12.5	12.4	12.6	7.3	18.7	9.8	6.0	11.5	11.5	11.5	8.5	15.9	11.1	3.8	12.2	12.2	12.2
		10	11.3	53.9	14.5	34.3	21.4	21.4	21.5	8.6	27.2	11.9	9.3	13.7	13.6	13.7	8.1	18.5	11.3	4.5	12.3	12.2	12.3
		20	23.4	81.4	31.7	60.8	38.3	38.4	38.2	12.5	49.5	17.5	20.2	19.3	19.1	19.3	9.9	29.7	14.1	8.6	14.9	15.0	15.0
	1.5	3	13.5	53.4	16.2	34.8	21.7	21.7	21.7	11.4	32.5	14.4	14.7	16.2	16.3	16.3	12.0	26.6	14.9	10.3	16.1	16.2	15.9
		4	17.8	67.6	23.1	47.0	28.9	29.0	29.3	12.6	40.0	16.2	18.1	18.1	18.1	18.1	12.0	29.5	15.3	11.1	16.4	16.4	16.4
		5	22.6	77.7	29.9	56.8	36.1	36.2	35.6	14.0	47.9	18.3	22.2	20.0	20.1	20.2	12.3	33.8	16.2	12.9	17.1	17.1	17.2
		10	47.3	97.4	63.1	87.6	64.1	64.1	63.8	22.1	81.0	29.3	45.9	30.7	30.8	31.0	16.3	58.8	21.8	24.4	22.6	22.5	22.5
		20	82.1	100.0	94.8	99.4	91.3	91.3	91.4	40.1	99.1	50.7	82.8	51.0	51.1	51.2	26.5	93.3	34.6	55.0	35.0	34.9	34.9
3x3	0.75	3	4.3	35.1	5.0	18.2	7.7	7.6	8.3	7.3	27.4	10.8	9.0	10.4	10.6	13.2	10.0	24.5	14.5	6.1	13.0	13.0	16.2
		4	4.5	42.2	5.5	23.1	8.5	8.4	9.7	7.1	31.2	10.9	10.9	10.6	10.6	13.4	9.3	26.2	14.2	6.5	12.5	12.4	15.7
		5	4.9	48.6	6.2	28.4	9.7	9.7	11.3	7.5	35.6	11.4	13.0	10.9	10.8	13.9	8.9	28.6	13.9	7.4	12.0	12.2	15.5
		10	8.0	72.5	11.6	51.0	15.0	15.3	19.9	8.8	56.5	14.4	26.0	12.9	12.9	17.0	9.4	43.4	15.2	13.7	12.7	12.7	16.6
		20	15.7	93.3	27.2	80.5	27.0	27.2	38.5	12.9	83.9	21.3	54.4	18.2	18.2	24.2	11.6	71.6	18.7	33.4	15.3	15.2	20.2
	1.5	3	9.8	75.1	13.9	55.0	16.0	15.8	20.0	11.1	59.8	16.6	32.5	15.0	15.2	19.3	12.6	49.7	18.0	23.0	16.0	15.9	19.7
		4	12.8	86.1	19.4	69.1	21.0	20.9	27.8	12.4	72.0	19.0	43.0	17.0	17.2	21.9	12.9	59.5	19.0	28.8	16.3	16.5	20.7
		5	15.6	92.1	25.9	78.8	25.3	25.5	35.4	13.8	81.3	21.3	53.5	18.7	18.6	24.3	13.4	68.9	19.9	36.1	17.1	17.0	21.6
		10	34.5	99.7	60.5	97.7	49.2	49.2	69.1	22.5	98.4	34.4	87.8	29.3	29.2	37.6	18.3	94.7	27.7	70.8	22.9	22.9	29.4
		20	68.6	100.0	95.1	100.0	80.9	81.1	96.0	40.1	100.0	57.2	99.7	48.7	48.5	60.0	29.6	100.0	42.3	97.6	35.6	35.4	44.0
4x4	0.75	3	3.7	43.5	4.3	21.1	6.6	6.7	7.5	6.5	36.8	10.4	12.5	9.4	9.4	13.3	9.4	33.2	14.9	9.0	12.1	12.1	17.3

		4	4.1	51.2	4.6	26.6	7.4	7.5	8.8	6.6	43.0	10.9	16.0	9.6	9.5	13.8	9.0	37.7	14.8	10.8	11.8	11.8	17.1
		5	4.4	57.6	5.2	32.3	8.2	8.2	10.1	6.7	48.6	11.5	19.7	9.9	9.7	14.7	8.8	41.7	15.0	12.8	11.6	11.6	17.2
		10	6.3	80.4	9.1	57.6	11.6	11.8	17.5	8.1	72.0	14.6	39.5	11.7	11.7	18.1	9.3	63.0	16.1	26.3	12.1	12.1	18.4
		20	11.6	96.4	21.7	85.9	19.9	19.8	35.0	11.8	93.0	21.5	72.7	16.5	16.2	25.6	11.5	88.3	20.4	57.3	14.9	15.0	22.8
	1.5	3	7.7	83.5	11.2	63.0	12.4	12.5	17.4	10.0	75.6	16.3	47.1	13.7	13.8	19.7	12.1	67.6	18.9	35.7	15.3	15.1	21.4
		4	9.7	91.7	15.7	76.5	15.9	15.8	24.5	11.3	85.7	18.9	60.1	15.4	15.3	22.8	12.6	78.5	20.3	46.8	15.8	15.9	22.8
		5	11.7	95.9	20.9	85.1	19.2	19.3	31.9	12.3	92.1	21.5	71.5	16.7	16.9	25.4	13.1	86.4	21.5	57.2	16.5	16.4	24.1
		10	24.9	99.9	53.0	99.0	36.9	36.6	66.2	20.2	99.7	34.8	96.5	26.4	26.3	39.5	17.7	99.1	29.2	90.6	21.9	22.1	31.8
		20	54.2	100.0	92.9	100.0	67.4	67.4	95.8	36.3	100.0	57.6	100.0	43.9	44.1	62.2	28.9	100.0	44.6	99.9	34.2	34.0	47.4
2x4	0.75	3	5.0	23.9	5.6	17.5	7.4	8.2	8.2	6.8	19.5	10.5	8.0	9.0	12.5	12.5	8.7	20.1	14.0	5.6	10.9	15.6	15.6
		4	5.4	30.4	6.6	22.5	8.2	9.6	9.8	6.5	21.8	10.7	9.7	8.8	12.6	12.6	8.0	20.2	13.3	6.0	10.3	14.8	14.7
		5	5.8	36.5	7.3	27.1	8.8	11.2	11.0	6.6	24.4	11.0	11.5	8.9	13.1	13.1	7.5	20.8	13.1	6.4	9.8	14.5	14.6
		10	8.9	62.9	13.4	49.1	13.1	19.4	19.0	7.7	39.7	13.7	22.9	10.5	15.9	15.8	7.7	28.0	13.9	11.7	10.0	15.3	15.3
		20	15.7	89.8	29.1	78.6	21.8	37.0	37.0	10.9	68.6	20.0	49.4	14.5	22.5	22.5	9.5	48.9	17.7	27.9	12.1	19.3	19.3
	1.5	3	10.1	63.0	15.0	52.7	14.1	19.7	19.8	9.9	43.8	15.9	30.0	12.7	18.3	18.2	11.0	36.0	17.3	20.5	13.5	18.9	19.0
		4	12.7	77.6	20.8	66.3	17.7	26.8	26.9	10.9	54.9	18.2	39.1	14.0	20.6	20.6	10.8	42.1	17.8	25.6	13.5	19.5	19.5
		5	15.4	87.4	27.3	76.9	21.2	34.1	34.3	11.8	64.8	20.4	48.8	15.3	23.0	23.0	11.2	49.3	18.9	31.5	13.9	20.7	20.7
		10	29.9	99.5	59.3	97.2	38.2	65.7	66.0	18.4	93.9	32.3	84.3	23.0	35.2	35.3	14.7	81.0	25.8	64.2	18.2	27.6	27.6
		20	56.4	100.0	93.0	100.0	65.2	94.1	94.2	33.0	100.0	54.5	99.4	39.3	57.3	57.4	23.5	99.3	39.6	95.9	28.3	42.0	41.7

Table 23. Test power estimates when samples are taken from Beta (10,10) distributions

			$\sigma_{11}^2:\sigma_{12}^2 \ldots :\sigma_{ij}^2$																				
			1:1:…:1							1:1:…:10							1:1:…:20						
rxc	δ	n	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB	A	B	AxB	Cont.	A	B	AxB
2x2	0.75	3	4.9	22.3	5.6	12.5	9.1	9.0	9.1	7.8	16.3	9.7	5.3	11.5	11.5	11.5	10.0	16.8	12.5	4.5	13.5	13.6	13.6
		4	5.5	27.0	6.4	15.5	10.7	10.8	10.8	7.3	17.2	9.6	5.6	11.3	11.4	11.3	8.8	15.6	11.4	3.9	12.5	12.5	12.5
		5	6.4	32.0	7.4	18.7	12.5	12.3	12.5	7.4	18.3	9.9	5.8	11.6	11.6	11.7	8.4	15.8	11.1	3.8	12.2	12.2	12.1
		10	11.2	54.0	14.4	34.4	21.1	21.6	21.3	8.7	27.4	11.9	9.5	13.8	13.6	13.7	8.0	18.4	11.2	4.5	12.2	12.3	12.1
		20	23.3	81.1	31.7	60.2	38.4	38.1	38.0	12.5	49.4	17.3	20.1	19.3	19.3	19.2	9.9	29.7	14.1	8.5	14.8	15.0	15.0
	1.5	3	13.1	53.1	16.0	34.7	21.3	21.3	21.2	11.5	32.3	14.6	14.7	16.1	16.3	16.4	12.5	26.5	15.2	10.1	16.3	16.4	16.3
		4	17.7	67.1	22.5	46.7	28.9	28.6	28.7	12.5	39.5	16.1	17.8	17.9	18.0	18.0	12.2	29.3	15.5	11.1	16.5	16.6	16.6
		5	22.3	77.8	29.3	56.9	35.6	35.5	35.4	13.9	47.1	18.1	21.7	19.9	19.8	20.0	12.3	32.9	16.0	12.4	16.9	16.9	16.9
		10	47.2	97.5	63.2	87.5	64.0	63.7	64.0	22.0	80.9	29.0	45.5	30.7	30.5	30.4	16.2	58.4	21.6	23.7	22.3	22.3	22.3
		20	82.1	100.0	94.9	99.4	91.5	91.4	91.4	40.0	99.2	50.8	82.9	51.1	50.8	51.2	26.1	93.6	34.3	54.8	34.6	34.6	34.7
3x3	0.75	3	4.2	34.9	5.0	18.1	7.6	7.5	8.3	7.5	27.0	11.0	8.7	10.8	10.6	13.2	10.3	24.7	15.0	6.0	13.3	13.3	16.7
		4	4.5	42.0	5.4	23.4	8.6	8.5	9.7	7.4	30.9	11.2	10.5	10.6	10.7	13.6	9.4	26.2	14.2	6.5	12.4	12.6	15.8
		5	4.8	48.4	6.1	28.0	9.5	9.6	11.2	7.4	35.2	11.5	12.5	10.8	10.9	13.9	9.2	28.5	14.2	7.3	12.2	12.3	15.8
		10	7.8	72.7	11.5	50.9	14.9	15.1	19.8	8.8	56.1	14.4	25.8	12.9	12.8	17.0	9.4	43.0	15.2	13.7	12.6	12.5	16.6
		20	15.7	93.3	27.3	80.1	27.0	27.0	38.7	12.8	84.1	20.9	54.2	18.0	18.0	23.8	11.7	71.6	19.0	33.2	15.4	15.4	20.5
	1.5	3	9.6	75.0	13.7	55.0	15.8	16.1	19.9	11.1	59.3	16.4	32.0	15.1	15.1	19.0	12.9	49.3	18.3	22.2	16.3	16.5	20.0
		4	12.5	85.9	18.9	68.3	20.5	20.6	27.2	12.4	71.6	18.8	42.9	16.9	16.9	21.6	12.9	58.9	18.9	28.3	16.4	16.6	20.7
		5	15.4	92.2	25.5	78.4	25.3	25.4	35.1	13.9	81.2	21.1	53.2	18.9	18.7	24.1	13.5	68.5	20.1	35.2	17.1	17.3	21.7
		10	34.3	99.7	60.1	97.8	49.1	48.9	68.8	22.2	98.4	34.1	87.9	28.9	28.8	37.3	18.2	94.7	27.2	70.5	22.9	22.8	29.0
		20	68.7	100.0	95.2	100.0	81.0	80.9	96.0	40.0	100.0	57.0	99.7	48.3	48.6	59.8	29.2	100.0	42.1	97.8	35.2	35.1	43.9
4x4	0.75	3	3.8	43.2	4.3	21.1	6.8	6.8	7.6	6.8	36.8	10.6	12.5	9.7	9.6	13.5	9.8	33.1	15.4	8.7	12.6	12.4	17.8
		4	4.1	50.8	4.6	26.7	7.5	7.4	8.9	6.6	42.9	10.9	16.1	9.7	9.5	14.0	9.2	37.1	14.9	10.3	12.0	11.7	17.2
		5	4.2	57.6	5.0	32.5	8.0	8.0	10.0	6.9	48.9	11.5	19.8	10.0	10.1	14.7	9.0	41.5	15.1	12.4	11.7	11.7	17.3
		10	6.3	80.3	8.9	57.3	11.7	11.8	17.2	8.0	71.9	14.3	39.4	11.7	11.5	18.0	9.2	62.7	16.1	26.3	12.0	12.0	18.2
		20	11.6	96.4	21.5	86.0	19.8	19.9	34.6	11.7	93.2	21.4	72.9	16.3	16.4	25.6	11.6	88.4	20.5	57.3	15.0	15.0	22.9
	1.5	3	7.7	83.3	11.3	62.9	12.5	12.5	17.5	10.0	75.3	16.4	46.6	13.7	13.7	19.7	12.4	67.3	19.2	35.6	15.5	15.3	21.6
		4	9.5	91.8	15.4	76.2	15.8	15.7	24.1	11.2	85.7	18.7	59.9	15.3	15.1	22.4	12.6	78.1	20.0	46.1	15.8	15.6	22.4
		5	11.6	96.0	20.5	85.3	19.1	19.1	31.3	12.3	92.2	21.3	71.6	16.7	16.8	25.4	13.0	86.3	21.4	57.1	16.4	16.4	23.9
		10	25.0	99.9	52.7	99.0	36.9	36.7	66.0	20.0	99.7	34.5	96.6	25.9	26.2	39.4	17.7	99.1	29.0	90.9	21.8	21.8	31.6
		20	54.0	100.0	93.1	100.0	67.2	67.5	96.0	36.4	100.0	57.9	100.0	44.2	44.1	62.6	28.8	100.0	44.4	99.9	34.0	33.9	47.2
2x4	0.75	3	4.9	23.9	5.5	17.6	7.3	8.0	8.1	6.8	19.7	10.8	8.0	9.1	12.8	12.7	8.8	20.0	14.0	5.6	11.0	15.6	15.6
		4	5.3	30.2	6.2	22.1	8.0	9.7	9.4	6.7	22.0	10.9	9.6	9.1	13.0	12.9	8.1	20.1	13.4	5.9	10.2	14.8	14.9
		5	5.9	36.4	7.4	27.1	8.9	11.3	11.2	6.7	24.4	11.2	11.3	9.1	13.3	13.3	7.6	21.1	13.4	6.3	9.8	14.9	14.9
		10	9.1	62.4	13.4	48.8	13.2	19.3	19.1	7.9	39.6	13.7	22.8	10.7	16.1	16.0	7.5	27.4	13.6	11.1	9.8	15.2	15.1
		20	15.7	89.9	29.3	78.7	21.9	36.7	36.9	10.7	68.6	19.8	49.7	14.2	22.3	22.4	9.3	48.4	17.4	27.7	12.1	18.9	19.0
	1.5	3	10.1	62.4	14.8	52.4	13.9	19.5	19.7	10.0	42.8	15.6	29.1	12.7	18.0	17.9	11.1	35.5	17.3	19.9	13.6	19.0	19.1
		4	12.7	77.5	20.5	66.0	17.5	26.6	26.5	10.7	54.2	18.1	39.0	13.9	20.5	20.6	10.9	41.4	17.8	25.0	13.4	19.5	19.6
		5	15.2	87.1	26.6	76.6	20.8	33.7	33.5	11.9	64.7	20.3	48.5	15.4	22.9	22.8	11.0	48.4	18.8	31.0	13.7	20.6	20.5
		10	29.6	99.5	59.3	97.1	37.8	65.5	65.7	18.5	94.0	32.3	84.3	23.1	35.4	35.2	14.7	80.8	25.4	63.8	18.1	27.2	27.3
		20	56.8	100.0	92.9	100.0	65.5	94.2	94.1	32.7	100.0	54.3	99.4	38.9	57.3	57.2	23.4	99.4	39.6	96.0	28.1	41.7	41.7

Table 24. Test power estimates when samples are taken from Beta (10,5) distributions

			$\sigma_{11}^2:\sigma_{12}^2 \ldots :\sigma_{ij}^2$																			
			1:1:…:1							1:1:…:10							1:1:…:20					
rxc	δ	n	A	B	AxB	***Cont.***	***A***	***B***	***AxB***	A	B	AxB	***Cont.***	***A***	***B***	***AxB***	***A***	B	AxB	***Cont.***	***A***	***B***	***AxB***
2x2	0.75	3	4.9	21.6	5.6	***12.4***	***9.0***	***9.0***	***9.0***	7.2	16.4	9.1	***6.1***	***10.8***	***10.8***	***10.8***	9.2	16.6	11.7	***5.2***	***12.8***	***12.8***	***12.9***
		4	5.5	26.5	6.5	***15.4***	***10.9***	***10.7***	***10.8***	6.7	17.2	8.7	***6.4***	***10.5***	***10.5***	***10.5***	8.3	16.2	10.8	***4.9***	***11.9***	***11.8***	***11.9***
		5	6.3	31.5	7.5	***18.5***	***12.6***	***12.5***	***12.6***	6.7	18.3	8.9	***6.7***	***10.8***	***10.6***	***10.7***	7.4	15.7	10.0	***4.7***	***11.1***	***11.2***	***11.1***
		10	11.2	53.2	14.4	***33.8***	***21.3***	***21.0***	***21.2***	7.7	27.0	10.7	***10.4***	***12.6***	***12.5***	***12.5***	7.1	18.2	10.3	***5.4***	***11.2***	***11.1***	***11.2***
		20	23.4	81.5	32.0	***60.0***	***38.4***	***38.2***	***38.5***	11.6	49.1	16.3	***20.7***	***18.3***	***18.3***	***18.2***	9.0	29.4	12.9	***9.7***	***14.0***	***13.8***	***13.7***
	1.5	3	13.3	52.7	16.2	***34.1***	***21.6***	***21.5***	***21.4***	10.3	31.8	12.9	***15.7***	***14.7***	***14.7***	***14.7***	11.2	26.3	13.8	***11.6***	***14.7***	***14.8***	***14.9***
		4	17.7	67.4	22.7	***46.2***	***28.9***	***28.6***	***28.9***	11.2	39.3	14.3	***19.3***	***16.3***	***16.3***	***16.2***	10.7	29.2	13.7	***12.5***	***14.8***	***15.0***	***14.8***
		5	22.4	78.0	29.5	***56.5***	***35.6***	***35.6***	***35.6***	12.3	47.1	16.2	***23.4***	***18.1***	***17.9***	***18.1***	10.7	32.8	14.0	***14.2***	***15.1***	***15.1***	***15.1***
		10	47.3	97.7	63.4	***87.7***	***63.9***	***64.1***	***64.1***	20.7	81.0	27.7	***45.9***	***29.5***	***29.2***	***29.4***	14.3	58.1	19.5	***25.6***	***20.5***	***20.4***	***20.4***
		20	82.1	100.0	95.0	***99.5***	***91.3***	***91.2***	***91.4***	39.2	99.3	50.1	***82.0***	***50.7***	***50.9***	***50.8***	25.1	93.7	33.3	***54.7***	***33.9***	***33.8***	***33.7***
3x3	0.75	3	4.2	33.5	4.7	***17.8***	***7.6***	***7.5***	***8.2***	7.0	26.4	10.2	***9.4***	***10.0***	***10.1***	***12.4***	9.7	24.4	14.2	***6.7***	***12.6***	***12.7***	***15.8***
		4	4.5	41.0	5.5	***22.9***	***8.5***	***8.5***	***9.7***	6.9	30.4	10.2	***11.3***	***10.1***	***10.0***	***12.6***	8.7	25.8	13.4	***7.4***	***11.7***	***11.6***	***15.1***
		5	5.0	47.7	6.2	***27.8***	***9.6***	***9.5***	***11.4***	6.7	34.8	10.6	***13.5***	***10.0***	***10.0***	***12.9***	8.6	27.9	13.3	***8.1***	***11.4***	***11.4***	***14.8***
		10	7.9	72.6	11.4	***50.3***	***15.0***	***15.1***	***19.7***	8.2	55.9	13.4	***26.1***	***12.2***	***12.0***	***15.8***	8.5	42.6	14.2	***14.4***	***11.7***	***11.8***	***15.7***
		20	15.6	93.8	27.2	***80.2***	***26.9***	***27.0***	***38.6***	12.3	84.2	20.2	***54.1***	***17.4***	***17.4***	***23.0***	10.8	71.3	17.7	***33.3***	***14.4***	***14.4***	***19.2***
	1.5	3	9.7	74.9	13.7	***53.7***	***15.9***	***15.9***	***19.8***	10.1	58.9	14.8	***32.6***	***13.9***	***13.9***	***17.3***	11.6	48.8	16.7	***23.2***	***14.8***	***14.8***	***18.5***
		4	12.3	86.7	19.1	***68.0***	***20.5***	***20.6***	***27.5***	11.4	71.6	16.9	***42.8***	***15.6***	***15.4***	***19.5***	11.7	58.8	17.2	***29.0***	***15.1***	***14.9***	***18.9***
		5	15.6	93.1	25.4	***78.5***	***25.5***	***25.4***	***35.0***	12.7	81.4	19.6	***52.8***	***17.4***	***17.3***	***22.5***	12.1	68.1	18.4	***35.8***	***15.7***	***15.7***	***20.0***
		10	34.2	99.8	60.4	***98.0***	***49.1***	***48.7***	***69.2***	20.9	98.7	32.8	***87.4***	***27.7***	***27.7***	***36.1***	16.9	94.9	25.9	***69.5***	***21.3***	***21.3***	***27.6***
		20	68.8	100.0	95.2	***100.0***	***81.2***	***80.9***	***96.0***	39.5	100.0	57.0	***99.7***	***48.1***	***48.3***	***59.9***	28.4	100.0	41.3	***97.4***	***34.4***	***34.4***	***43.0***
4x4	0.75	3	3.7	41.9	4.2	***20.6***	***6.6***	***6.7***	***7.5***	6.4	36.2	10.1	***13.2***	***9.2***	***9.3***	***12.8***	9.2	32.6	14.7	***9.5***	***12.0***	***12.0***	***16.9***
		4	4.1	49.9	4.5	***26.2***	***7.4***	***7.4***	***8.6***	6.3	41.8	10.4	***16.2***	***9.3***	***9.4***	***13.3***	8.6	36.6	14.1	***11.0***	***11.3***	***11.4***	***16.4***
		5	4.4	57.2	5.0	***32.0***	***8.2***	***8.1***	***9.8***	6.1	47.6	10.5	***19.7***	***9.2***	***9.1***	***13.4***	8.3	40.9	13.9	***13.1***	***11.0***	***10.8***	***16.2***
		10	6.2	80.6	8.9	***56.7***	***11.8***	***11.6***	***17.4***	7.5	71.7	13.5	***39.1***	***11.1***	***11.1***	***17.0***	8.5	62.6	15.4	***26.4***	***11.4***	***11.4***	***17.6***
		20	11.4	96.7	21.5	***86.1***	***19.7***	***19.6***	***34.7***	11.2	93.6	20.7	***72.4***	***15.7***	***15.9***	***24.8***	11.1	88.3	19.5	***57.0***	***14.3***	***14.2***	***21.9***
	1.5	3	7.8	84.1	11.2	***62.1***	***12.5***	***12.4***	***17.3***	9.2	75.3	14.8	***45.8***	***12.7***	***12.6***	***17.8***	11.2	66.9	17.5	***35.5***	***14.2***	***14.1***	***19.9***
		4	9.6	92.8	15.5	***75.9***	***15.7***	***15.6***	***24.3***	10.1	86.2	17.0	***59.4***	***14.0***	***14.1***	***20.6***	11.4	78.3	18.4	***46.1***	***14.5***	***14.4***	***20.8***
		5	11.8	96.6	20.6	***85.2***	***19.1***	***19.1***	***31.5***	11.6	92.6	19.6	***70.9***	***15.8***	***15.7***	***23.5***	11.9	86.7	19.7	***56.3***	***15.1***	***15.1***	***22.2***
		10	24.9	100.0	52.9	***99.2***	***36.8***	***36.9***	***66.0***	19.0	99.8	33.3	***96.7***	***25.0***	***25.2***	***38.2***	16.5	99.4	27.6	***90.6***	***20.5***	***20.5***	***30.2***
		20	54.1	100.0	93.1	***100.0***	***67.3***	***67.4***	***96.1***	36.0	100.0	57.7	***100.0***	***43.8***	***43.7***	***62.3***	28.0	100.0	43.9	***99.9***	***33.2***	***33.3***	***46.7***
2x4	0.75	3	4.8	22.9	5.7	***17.1***	***7.1***	***8.2***	***8.4***	6.6	19.0	9.9	***8.8***	***8.7***	***11.7***	***11.8***	8.5	19.5	13.2	***6.4***	***10.6***	***14.6***	***14.7***
		4	5.3	29.6	6.6	***21.9***	***8.1***	***9.7***	***9.9***	6.2	21.2	9.9	***10.2***	***8.5***	***12.0***	***11.9***	7.5	19.8	12.6	***6.6***	***9.6***	***14.2***	***14.0***
		5	5.9	36.1	7.5	***26.7***	***8.9***	***11.3***	***11.3***	6.2	23.6	10.3	***12.1***	***8.6***	***12.1***	***12.2***	7.2	20.5	12.4	***7.2***	***9.2***	***13.9***	***13.9***
		10	9.0	62.3	13.6	***48.4***	***13.3***	***19.2***	***19.6***	7.2	39.0	12.8	***23.4***	***9.9***	***15.0***	***15.0***	6.9	27.4	13.1	***12.4***	***9.1***	***14.5***	***14.6***
		20	15.8	90.0	29.3	***78.3***	***22.0***	***36.8***	***36.9***	10.4	69.0	19.0	***49.1***	***13.8***	***21.7***	***21.6***	8.6	48.8	16.4	***28.8***	***11.2***	***18.2***	***17.9***
	1.5	3	10.0	62.4	14.7	***51.8***	***13.9***	***19.4***	***19.5***	9.2	42.3	14.5	***29.8***	***11.7***	***16.5***	***16.5***	10.2	34.9	16.0	***21.0***	***12.6***	***17.6***	***17.6***
		4	12.6	77.7	20.9	***65.6***	***17.5***	***26.7***	***27.1***	9.8	53.7	16.4	***38.8***	***12.8***	***18.7***	***18.7***	9.8	40.9	16.2	***26.2***	***12.3***	***17.9***	***17.8***
		5	15.3	87.7	26.9	***76.4***	***21.1***	***33.7***	***33.9***	10.9	64.7	18.5	***48.4***	***14.2***	***21.0***	***21.0***	9.9	47.9	17.1	***31.8***	***12.6***	***18.6***	***18.6***
		10	29.9	99.6	59.0	***97.5***	***38.1***	***65.7***	***65.5***	17.2	94.5	30.9	***83.6***	***21.8***	***34.1***	***33.9***	13.4	81.8	24.0	***63.6***	***16.8***	***26.0***	***25.9***
		20	56.6	100.0	93.1	***100.0***	***65.3***	***94.3***	***94.1***	32.1	100.0	54.3	***99.4***	***38.6***	***57.2***	***57.3***	22.4	99.6	38.7	***95.4***	***27.2***	***40.9***	***40.8***

Table 25. Test power estimates when samples are taken from Beta (5,10) distributions

			$\sigma_{11}^2:\sigma_{12}^2 \ldots :\sigma_{ij}^2$																			
			1:1:…:1							1:1:…:10							1:1:…:20					
rxc	δ	n	A	B	AxB	***Cont.***	***A***	***B***	***AxB***	A	B	AxB	***Cont.***	***A***	***B***	***AxB***	***A***	B	AxB	***Cont.***	***A***	***B***	***AxB***
2x2	0.75	3	4.9	22.8	5.6	***12.4***	***9.0***	***9.1***	***9.1***	8.8	16.7	10.9	***4.8***	***12.8***	***12.7***	***12.7***	10.9	16.9	13.6	***3.9***	***14.7***	***14.7***	***14.7***
		4	5.5	28.1	6.4	***15.8***	***10.7***	***10.8***	***10.8***	8.3	17.4	10.8	***4.6***	***12.5***	***12.6***	***12.5***	10.0	16.2	12.8	***3.3***	***13.8***	***13.7***	***13.8***
		5	6.4	32.7	7.5	***18.9***	***12.5***	***12.4***	***12.5***	8.4	18.9	11.1	***5.2***	***12.7***	***12.9***	***12.8***	9.5	16.0	12.3	***2.8***	***13.3***	***13.3***	***13.3***
		10	11.1	53.9	14.6	***34.6***	***21.2***	***21.2***	***21.5***	9.7	27.7	13.1	***8.4***	***14.8***	***15.0***	***14.7***	9.1	18.5	12.4	***3.5***	***13.4***	***13.3***	***13.4***
		20	23.4	81.1	31.7	***60.9***	***38.3***	***38.2***	***38.3***	13.4	49.8	18.4	***19.2***	***20.2***	***20.2***	***20.2***	10.8	29.8	15.2	***7.3***	***15.9***	***16.1***	***16.0***
	1.5	3	13.2	53.9	16.2	***35.4***	***21.5***	***21.6***	***21.2***	13.2	32.5	16.4	***13.2***	***18.1***	***18.2***	***18.2***	13.9	26.8	17.0	***8.9***	***18.0***	***18.1***	***18.0***
		4	17.6	67.4	22.7	***47.4***	***28.8***	***28.8***	***29.0***	14.2	40.0	18.0	***16.4***	***19.8***	***19.9***	***19.7***	13.8	29.4	17.2	***9.4***	***18.4***	***18.0***	***18.0***
		5	22.4	77.5	29.5	***57.7***	***35.6***	***35.8***	***35.6***	15.6	47.8	20.0	***20.4***	***21.9***	***21.7***	***21.7***	14.1	33.4	18.2	***10.5***	***18.9***	***18.9***	***19.0***
		10	47.0	97.2	63.3	***87.4***	***63.8***	***64.1***	***64.2***	23.7	80.8	30.9	***45.3***	***32.2***	***32.0***	***32.2***	17.8	58.3	23.2	***21.9***	***23.8***	***23.9***	***23.8***
		20	82.2	100.0	94.9	***99.3***	***91.4***	***91.4***	***91.3***	40.8	99.1	50.9	***83.9***	***51.3***	***51.3***	***51.3***	27.7	93.5	35.4	***55.0***	***35.8***	***36.0***	***35.7***
3x3	0.75	3	4.1	36.1	5.0	***18.3***	***7.6***	***7.6***	***8.3***	8.0	28.0	11.9	***8.2***	***11.4***	***11.5***	***14.4***	11.2	25.3	15.9	***5.4***	***14.2***	***14.3***	***17.7***
		4	4.5	42.8	5.4	***23.6***	***8.7***	***8.6***	***9.6***	7.9	31.7	12.0	***10.0***	***11.5***	***11.6***	***14.5***	10.5	26.8	15.4	***5.7***	***13.6***	***13.6***	***17.1***
		5	5.0	48.9	6.0	***28.5***	***9.8***	***9.5***	***11.1***	8.1	35.9	12.8	***12.1***	***11.8***	***11.7***	***15.3***	10.2	29.3	15.4	***6.5***	***13.3***	***13.5***	***16.8***
		10	7.9	72.5	11.6	***51.4***	***15.0***	***15.1***	***19.9***	9.7	56.8	15.4	***25.5***	***14.0***	***13.9***	***18.1***	10.1	43.7	16.2	***12.6***	***13.3***	***13.6***	***17.7***
		20	15.9	93.0	27.2	***80.2***	***27.2***	***26.9***	***38.7***	13.6	84.0	22.0	***55.2***	***18.9***	***19.0***	***25.0***	12.4	71.6	20.0	***32.9***	***16.2***	***16.4***	***21.5***

	1.5	3	9.7	74.7	13.8	55.5	16.0	15.9	19.9	12.3	59.9	18.3	32.2	16.5	16.7	21.0	14.3	49.8	20.3	21.3	18.0	18.0	22.1
		4	12.6	85.3	19.1	69.0	20.7	20.7	27.6	13.6	71.7	20.3	43.3	18.1	18.3	23.2	14.4	59.4	20.8	27.6	18.1	18.0	22.5
		5	15.8	91.8	25.7	79.1	25.6	25.8	35.5	15.0	81.0	23.2	53.8	20.2	20.4	26.1	15.2	68.9	22.1	34.7	19.0	19.0	23.8
		10	34.5	99.6	60.1	97.5	49.1	48.9	68.9	23.3	98.1	35.5	88.0	30.1	30.1	38.6	19.5	94.4	28.6	71.5	24.0	24.2	30.3
		20	68.6	100.0	95.0	100.0	80.8	80.7	95.9	40.5	100.0	57.2	99.7	48.7	48.7	59.9	30.3	100.0	42.7	98.0	36.1	36.2	44.3
4x4	0.75	3	3.7	44.5	4.3	21.4	6.7	6.7	7.5	7.0	37.5	11.6	12.0	10.1	10.2	14.6	10.3	33.9	16.3	8.2	13.2	13.3	18.7
		4	4.0	51.9	4.6	27.5	7.4	7.4	8.7	7.0	43.8	11.9	15.7	10.2	10.2	15.1	9.7	38.0	15.9	9.8	12.6	12.8	18.1
		5	4.2	58.2	5.1	33.3	8.1	7.9	10.1	7.2	49.2	12.4	19.2	10.5	10.6	15.7	9.6	42.7	16.1	11.9	12.5	12.6	18.3
		10	6.2	79.8	9.0	57.7	11.8	11.7	17.3	8.5	72.0	15.3	40.2	12.4	12.2	18.9	10.0	63.4	17.2	26.1	13.0	13.0	19.4
		20	11.6	96.1	21.6	86.1	20.0	20.1	34.7	12.2	92.9	22.2	73.2	16.8	16.9	26.4	12.5	88.2	21.2	57.8	15.9	15.9	23.6
	1.5	3	7.7	82.8	11.3	63.8	12.6	12.5	17.6	10.7	75.3	17.7	47.2	14.5	14.6	21.1	13.5	67.8	20.9	35.9	16.8	16.9	23.3
		4	9.6	90.9	15.4	76.4	15.8	15.8	24.3	12.0	85.2	20.3	61.0	16.3	16.2	23.9	13.9	78.1	22.0	46.6	17.4	17.3	24.6
		5	11.6	95.4	20.7	85.4	19.1	19.4	31.5	13.4	91.5	22.7	71.9	17.9	17.9	26.7	14.5	86.2	23.2	57.8	17.9	17.9	25.6
		10	25.0	99.9	53.1	98.9	37.1	36.6	66.2	21.1	99.6	35.8	96.4	27.1	27.1	40.3	18.8	98.9	30.5	91.1	23.0	23.1	33.1
		20	54.1	100.0	92.8	100.0	67.3	67.5	95.8	37.1	100.0	57.5	100.0	44.6	44.3	61.8	29.6	100.0	44.9	99.9	34.6	34.8	47.6
2x4	0.75	3	4.8	24.4	5.5	17.6	7.2	8.2	8.1	7.4	20.4	11.7	7.4	9.8	13.8	13.8	9.6	20.9	15.4	4.9	12.0	16.9	17.0
		4	5.3	30.8	6.5	22.6	8.0	9.6	9.7	7.4	22.8	11.9	8.9	9.8	14.1	14.1	8.8	20.9	14.5	5.0	11.1	16.0	16.1
		5	5.8	36.9	7.5	27.3	8.9	11.1	11.2	7.3	25.0	12.2	10.6	9.8	14.3	14.2	8.5	21.6	14.4	5.5	10.7	15.9	15.8
		10	9.0	62.5	13.7	49.3	13.3	19.3	19.4	8.4	40.0	14.8	22.7	11.2	17.0	17.1	8.3	28.2	15.0	10.6	10.7	16.3	16.4
		20	15.8	89.6	29.1	78.8	21.9	37.0	36.9	11.7	68.5	20.9	49.7	15.2	23.4	23.5	10.0	48.7	18.1	27.4	12.7	19.9	19.7
	1.5	3	10.1	62.8	14.8	53.3	14.0	19.8	19.5	11.0	43.9	17.8	28.9	13.9	20.1	20.1	12.2	36.2	19.0	18.6	14.8	20.7	20.8
		4	12.6	77.3	20.6	66.8	17.5	26.9	26.7	12.1	54.6	19.9	38.8	15.3	22.4	22.3	12.3	42.1	19.7	23.9	15.1	21.5	21.4
		5	15.4	86.7	27.0	76.9	21.0	34.3	34.1	12.9	64.5	21.9	48.9	16.4	24.4	24.4	12.5	48.9	20.7	30.2	15.3	22.5	22.5
		10	29.7	99.4	59.2	97.0	38.0	65.6	65.7	19.5	93.5	33.5	84.6	24.1	36.5	36.4	16.1	80.4	27.0	64.8	19.5	28.9	28.9
		20	56.7	100.0	93.1	100.0	65.6	93.9	94.3	33.6	99.9	54.7	99.4	39.7	57.4	57.4	24.6	99.2	40.3	96.4	29.3	42.2	42.2

Table 26. Test power estimates when samples are taken from Chi-Sq (3) distributions

			$\sigma_{11}^2 : \sigma_{12}^2 \ldots : \sigma_{ij}^2$																				
			1:1:…:1							1:1:…:10							1:1:…:20						
rxc	δ	n	A	B	AxB	**Cont.**	***A***	***B***	***AxB***	A	B	AxB	**Cont.**	***A***	***B***	***AxB***	***A***	B	AxB	**Cont.**	***A***	***B***	***AxB***
2x2	0.75	3	5.5	21.8	6.3	13.9	9.8	9.6	9.8	5.4	17.5	6.5	9.8	8.4	8.3	8.3	8.1	19.1	10.0	9.5	11.4	11.4	11.4
		4	6.2	27.7	7.6	17.4	11.6	11.8	11.9	4.8	18.8	6.2	10.8	8.1	8.0	8.1	7.0	19.0	9.1	9.8	10.4	10.3	10.3
		5	7.2	33.0	8.8	20.4	13.6	13.7	13.7	4.6	20.4	6.3	11.6	8.0	8.1	8.2	6.3	18.9	8.6	9.7	9.8	9.8	9.8
		10	12.4	54.8	16.0	34.5	22.7	22.6	22.6	5.4	29.1	7.8	16.0	9.9	9.8	9.8	5.3	21.2	7.9	11.2	9.1	9.1	9.1
		20	24.8	82.3	33.6	60.2	39.6	39.1	39.6	9.2	50.7	13.4	26.0	15.8	15.6	15.6	6.5	31.7	10.2	15.7	11.4	11.4	11.4
	1.5	3	16.7	57.5	20.6	38.3	25.7	25.7	25.7	6.8	36.1	8.6	23.6	10.3	10.3	10.4	7.7	30.7	9.6	19.7	10.8	10.8	10.9
		4	21.6	71.1	27.4	49.5	32.9	33.2	32.9	7.5	44.2	9.9	28.0	12.0	11.7	11.9	7.0	34.2	9.3	22.1	10.7	10.5	10.6
		5	26.3	80.0	34.3	58.4	39.8	39.4	39.4	8.7	51.7	11.8	32.0	14.1	14.1	14.0	7.0	37.9	9.7	23.7	11.0	10.9	11.0
		10	49.8	97.6	65.2	87.6	65.6	66.1	65.5	17.2	81.6	24.1	50.8	26.7	26.7	26.6	10.3	61.3	15.2	34.7	16.5	16.5	16.4
		20	81.9	100.0	94.1	99.5	90.8	90.7	91.0	38.2	98.9	50.5	79.5	51.2	51.5	51.4	21.4	92.7	30.4	56.9	31.2	31.3	31.3
3x3	0.75	3	4.4	32.8	5.3	18.7	7.4	7.4	8.1	5.1	26.3	7.4	12.6	7.5	7.6	9.1	7.8	25.5	11.6	10.7	10.5	10.6	13.3
		4	4.8	41.2	6.0	23.5	8.8	8.6	9.7	4.9	30.7	7.4	14.8	7.5	7.6	9.4	7.0	27.7	10.9	11.7	9.8	9.8	12.6
		5	5.3	47.7	6.9	27.7	9.8	9.8	11.4	5.1	35.4	7.9	17.0	7.9	7.8	9.9	6.6	30.1	10.8	13.1	9.4	9.4	12.4
		10	8.5	74.1	12.6	49.4	15.5	15.4	20.4	6.3	56.1	10.3	28.8	9.8	9.7	12.7	6.4	43.9	11.4	20.0	9.3	9.2	12.9
		20	16.0	95.0	28.0	80.2	27.1	27.1	39.1	10.1	84.8	17.2	54.3	15.0	15.1	20.4	8.5	71.5	14.9	37.1	12.0	11.9	16.3
	1.5	3	10.9	78.2	16.1	54.2	16.8	17.0	21.7	7.2	62.1	10.4	37.5	10.1	10.1	12.4	8.0	52.6	11.6	30.7	10.6	10.6	13.2
		4	13.8	89.0	21.9	68.1	21.9	21.9	29.6	8.1	74.3	12.5	46.9	11.7	11.7	14.8	7.9	62.3	12.3	37.1	10.8	10.7	13.9
		5	17.1	94.7	28.5	78.6	26.7	27.2	37.5	9.5	83.4	15.3	55.9	13.8	13.9	18.1	8.2	70.9	13.6	43.0	11.4	11.5	15.3
		10	35.4	99.9	61.5	98.3	49.9	49.6	69.9	18.0	98.9	29.9	85.6	25.0	24.9	33.8	12.9	95.0	21.6	69.8	17.4	17.3	23.5
		20	69.2	100.0	94.8	100.0	81.5	81.4	95.9	38.2	100.0	57.4	99.4	48.0	47.8	61.1	25.4	100.0	39.5	95.1	31.9	31.8	41.7
4x4	0.75	3	4.0	40.7	4.9	20.5	6.5	6.6	7.5	5.0	34.6	7.6	15.2	7.1	7.1	9.6	7.3	32.3	11.6	12.5	9.8	9.6	13.8
		4	4.3	49.2	5.3	25.7	7.4	7.4	8.8	4.7	40.8	7.7	18.0	7.1	7.2	10.0	6.6	36.5	11.4	14.4	9.0	9.1	13.7
		5	4.5	56.8	5.9	30.7	8.0	8.0	10.1	5.0	47.4	8.1	21.3	7.5	7.5	10.6	6.4	41.5	11.5	16.9	8.9	8.9	13.7
		10	6.5	81.9	9.9	54.8	11.6	11.7	17.5	6.1	72.2	11.1	39.2	9.3	9.2	14.2	6.6	62.9	12.5	29.3	9.2	9.2	14.7
		20	11.7	97.7	22.4	86.4	19.9	19.9	35.1	9.5	94.7	18.1	71.6	13.8	13.9	22.1	8.9	89.5	17.0	56.9	12.0	12.0	19.4
	1.5	3	8.4	87.3	12.7	61.0	12.8	12.7	18.1	7.0	78.0	11.0	47.4	9.6	9.5	13.3	7.7	69.7	12.0	39.6	10.0	9.9	14.0
		4	10.4	94.9	17.4	75.5	16.2	16.0	25.3	7.9	88.4	13.3	60.0	11.0	11.1	16.3	7.9	80.3	13.3	49.6	10.5	10.5	15.4
		5	12.6	98.2	22.8	85.8	19.5	19.8	32.7	9.2	94.4	16.3	70.5	13.0	12.9	19.8	8.4	88.3	14.9	58.9	11.3	11.1	17.2
		10	25.5	100.0	53.7	99.6	37.1	37.2	66.3	16.9	99.9	31.1	96.2	22.9	23.0	36.5	13.2	99.5	24.0	88.8	17.2	17.4	26.8
		20	54.3	100.0	92.8	100.0	67.5	67.7	96.0	34.5	100.0	57.8	100.0	42.9	42.9	63.5	25.4	100.0	42.2	99.8	31.0	30.8	45.7
2x4	0.75	3	5.1	22.9	5.8	18.2	7.4	8.0	8.2	4.9	18.2	7.2	12.0	6.8	8.9	8.8	7.0	20.1	11.0	10.5	8.9	12.7	12.6
		4	5.6	30.0	6.7	22.7	8.3	9.9	9.8	4.8	21.0	7.2	14.0	6.7	8.9	9.0	6.0	20.6	10.4	11.5	8.1	11.9	11.9
		5	6.2	36.3	7.9	27.0	9.2	11.4	11.6	4.7	24.1	7.4	16.3	6.7	9.2	9.2	5.7	21.4	10.0	12.3	7.7	11.6	11.5
		10	9.3	63.2	14.4	47.6	13.5	19.7	20.0	5.6	39.4	10.0	26.7	8.1	12.0	12.1	5.3	28.7	10.3	18.2	7.4	12.0	11.8

		20	16.2	90.7	30.0	***78.4***	***22.5***	***37.3***	***37.7***	8.6	70.1	16.3	***50.2***	***12.0***	***18.8***	***19.0***	6.5	50.0	13.4	***33.1***	***9.1***	***15.2***	***15.1***
	1.5	3	11.3	65.5	17.0	***52.7***	***15.3***	***21.1***	***21.7***	6.7	44.4	9.9	***35.6***	***8.9***	***11.7***	***11.7***	6.9	36.9	10.9	***29.0***	***8.8***	***12.4***	***12.4***
		4	14.4	80.0	23.6	***66.0***	***19.2***	***28.8***	***29.4***	7.3	57.1	11.8	***44.2***	***9.9***	***13.8***	***14.0***	6.7	44.3	11.3	***34.4***	***8.8***	***12.8***	***12.9***
		5	16.8	88.7	30.0	***76.4***	***22.6***	***35.9***	***36.7***	8.3	68.1	14.2	***52.1***	***11.4***	***16.7***	***16.8***	6.8	52.0	12.1	***39.8***	***9.2***	***13.8***	***13.8***
		10	31.3	99.6	60.4	***97.7***	***39.4***	***66.6***	***66.6***	15.3	95.6	28.6	***81.8***	***20.1***	***31.8***	***32.1***	9.8	84.4	19.3	***65.0***	***13.1***	***21.6***	***21.4***
		20	57.3	100.0	92.5	***100.0***	***65.8***	***94.3***	***93.8***	30.8	100.0	54.8	***98.8***	***37.8***	***58.0***	***58.4***	19.3	99.6	36.5	***92.5***	***24.5***	***38.6***	***39.1***

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THE POISSON-WEIGHTED AKASH DISTRIBUTION AND ITS APPLICATIONS

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Abstract

A Poisson-Weighted Akash distribution which includes Poisson-Akash distribution has been proposed. Its moments and moments based statistical constants have been derived and studied. Maximum likelihood estimation has been discussed for estimating the parameters of the distribution. Finally, applications of the proposed distribution have been explained through two count datasets and the goodness of fit has been compared with other discrete distributions.

Keywords: *Weighted Akash distribution; Poisson- Akash distribution; Compounding; Moments; Skewness; Kurtosis; Maximum likelihood estimation; Applications*

1. Introduction

Shanker (2017) introduced the discrete Poisson- Akash distribution (PAD) to model count data defined by its probability mass function (pmf)

$$P_1(x;\theta)=\frac{\theta^3}{\theta^2+2}\cdot\frac{x^2+3x+\left(\theta^2+2\theta+3\right)}{\left(\theta+1\right)^{x+3}}\;;x=0,1,2,...,\theta>0 \qquad (1.1)$$

Moments and moments based measures, statistical properties; estimation of parameter using both the method of moment and the method of maximum likelihood and applications of PAD has been discussed by Shanker (2017). The PAD arises from the Poisson distribution when its parameter λ follows Akash distribution introduced by Shanker (2015) defined by its probability density function (pdf)

$$f_1(\lambda,\theta)=\frac{\theta^3}{\theta^2+2}\left(1+\lambda^2\right)e^{-\theta\lambda}\;;\lambda>0,\theta>0 \qquad (1.2)$$

The pdf (1.2) is a convex combination of exponential (θ) and gamma $(3,\theta)$ distributions. Shanker (2015) discussed statistical properties including moments based coefficients, hazard rate function, mean residual life function, mean deviations, stochastic ordering, Renyi entropy measure, order statistics, Bonferroni and Lorenz curves, stress- strength reliability, along with estimation of parameter and applications of Akash distribution to model lifetime data from biomedical science and engineering.

The first four moments about origin and the variance of PAD (1.1) obtained by Shanker (2017) are given by

$$\mu_1' = \frac{\theta^2+6}{\theta\left(\theta^2+2\right)}$$

$$\mu_2' = \frac{\theta^3+2\theta^2+6\theta+24}{\theta^2\left(\theta^2+2\right)}$$

$$\mu_3' = \frac{\theta^4+6\theta^3+12\theta^2+72\theta+120}{\theta^3\left(\theta^2+2\right)}$$

$$\mu_4' = \frac{\theta^5+14\theta^4+42\theta^3+192\theta^2+720\theta+720}{\theta^4\left(\theta^2+2\right)}$$

$$\mu_2 = \sigma^2 = \frac{\theta^5+\theta^4+8\theta^3+16\theta^2+12\theta+12}{\theta^2\left(\theta^2+2\right)^2}.$$

Sankaran (1970) proposed the Poisson-Lindley distribution (PLD) to model count data defined by its pmf

$$P_2\left(x;\theta\right) = \frac{\theta^2\left(x+\theta+2\right)}{\left(\theta+1\right)^{x+3}} \quad ; x=0,1,2,...,\theta>0 \tag{1.3}$$

Shanker and Hagos (2015) proposed a simple method of finding moments of PLD and discussed the applications of PLD to model count data from biological sciences. The PLD arises from the Poisson distribution when its parameter λ follows Lindley (1958) distribution defined by its probability density function (pdf)

$$f_2\left(\lambda;\theta\right) = \frac{\theta^2}{\theta+1}\left(1+\lambda\right)e^{-\theta\lambda} \; ; \lambda>0,\theta>0 \tag{1.4}$$

It can be easily verified that the pdf (1.4) is a convex combination of exponential (θ) and gamma $(2,\theta)$ distributions. Ghitany *et al* (2008) discussed statistical properties including moments based coefficients, hazard rate function, mean residual life function, mean deviations, stochastic ordering, Renyi entropy measure, order statistics, Bonferroni and Lorenz curves, stress- strength reliability, along with estimation of parameter and application of Lindley distribution to model waiting time data in a bank. Shanker *et al* (2015) have detailed study on modeling of various lifetime data from engineering and biomedical sciences using exponential and Lindley distribution and observed that there are many lifetime data where exponential distribution gives much better fit than Lindley distribution.

Ghitany *et al* (2011) introduced a two-parameter weighted Lindley distribution (WLD) having parameters θ and α and defined by its pdf

$$f_3(x;\theta,\alpha)=\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x)e^{-\theta x}\ ;x>0,\ \theta>0,\ \alpha>0 \tag{1.5}$$

where $\Gamma(\alpha)=\int_0^\infty e^{-y}\,y^{\alpha-1}\,dy;\ \alpha>0$ is the complete gamma function. Its structural properties including moments, hazard rate function, mean residual life function, estimation of parameters and applications for modeling survival time data has been discussed by Ghitany *et al* (2011). The corresponding cumulative distribution function (cdf) of WLD (1.5) is given by

$$F(x;\theta,\alpha)=1-\frac{(\theta+\alpha)\Gamma(\alpha,\theta x)+(\theta x)^{\alpha}e^{-\theta x}}{(\theta+\alpha)\Gamma(\alpha)}\ ;x>0,\ \theta>0,\ \alpha>0 \tag{1.6}$$

where

$$\Gamma(\alpha,z)=\int_z^\infty e^{-y}\,y^{\alpha-1}\,dy\,;\alpha>0,\ z\geq 0 \tag{1.7}$$

is the upper incomplete gamma function. It can be easily shown that at $\alpha=1$, WLD (1.5) reduces to Lindley (1958) distribution (1.4). Shanker *et al* (2016) discussed various moments based properties including coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of weighted Lindley distribution and its applications to model lifetime data from biomedical sciences and engineering. Shanker *et al* (2017) have proposed a three-parameter weighted Lindley distribution (TPWLD) which includes a two-parameter weighted Lindley distribution and one parameter Lindley distribution as particular cases and discussed its various structural properties, estimation of parameters and applications for modeling lifetime data from engineering and biomedical sciences.

Assuming that the parameter λ of the Poisson distribution follows WLD (1.5), El-Monsef and Sohsah (2014) proposed Poisson- weighted Lindley distribution (P-WLD) defined by its pmf

$$P_3(x;\theta,\alpha)=\frac{\Gamma(x+\alpha)}{\Gamma(x+1)\Gamma(\alpha)}\frac{\theta^{\alpha+1}}{(\theta+\alpha)}\frac{x+\theta+\alpha+1}{(\theta+1)^{x+\alpha+1}};x=0,1,2,...,\theta>0,\alpha>0 \tag{1.8}$$

It can be easily verified that PLD (1.3) is a particular case of P-WLD for $\alpha=1$.

Shanker and Shukla (2016) proposed a two-parameter weighted Akash distribution (WAD) having parameters θ and α and defined by its pdf

$$f_4(x;\theta,\alpha)=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)}\frac{x^{\alpha-1}}{\Gamma(\alpha)}(1+x^2)e^{-\theta x}\ ;x>0,\ \theta>0,\ \alpha>0 \tag{1.9}$$

Its structural properties including moments, hazard rate function, mean residual life function, estimation of parameters and applications for modeling survival time data has been discussed by Shanker and Shukla (2016). It can be easily shown that at $\alpha=1$, WAD (1.9) reduces to Akash distribution (1.2).

The main purpose of this paper is to introduce a two-parameter Poisson-Weighted Akash distribution, a Poisson mixture of two-parameter weighted Akash distribution proposed by Shanker and Shukla (2016). Its moments based measures including coefficients of variation, skewness, kurtosis and index of dispersion have been derived and their natures have been discussed graphically. The estimation of parameters has been discussed using the method of maximum likelihood. Applications and goodness of fit of the distribution has also been discussed through two examples of observed real count datasets and the fit has been compared with other discrete distributions.

2. The Poisson-weighted Akash distribution

Assuming that the parameter λ of the Poisson distribution follows WAD (1.9), the Poisson mixture of WAD can be obtained as

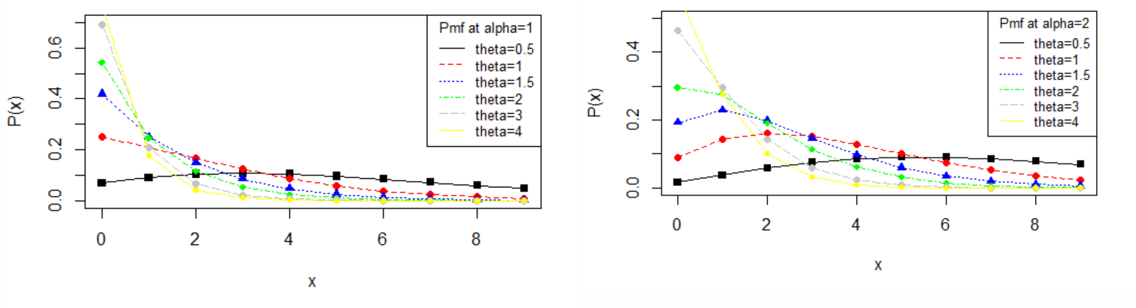
$$P_4(x;\theta,\alpha)=\int_0^\infty \frac{e^{-\lambda}\lambda^x}{\Gamma(x+1)}\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)}\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}(1+\lambda^2)e^{-\theta\lambda}d\lambda \quad (2.1)$$

$$=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)\Gamma(x+1)}\left[\int_0^\infty e^{-(\theta+1)\lambda}\lambda^{x+\alpha-1}d\lambda+\int_0^\infty e^{-(\theta+1)\lambda}\lambda^{x+\alpha+2-1}d\lambda\right]$$

$$=\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)\Gamma(x+1)}\left[\frac{\Gamma(x+\alpha)}{(\theta+1)^{x+\alpha}}+\frac{\Gamma(x+\alpha+2)}{(\theta+1)^{x+\alpha+2}}\right]$$

$$=\frac{\Gamma(x+\alpha)}{\Gamma(x+1)\Gamma(\alpha)}\frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)}\frac{x^2+(2\alpha+1)x+(\theta^2+\alpha^2+2\theta+\alpha+1)}{(\theta+1)^{x+\alpha+2}};x=0,1,2,...,\theta>0,\alpha>0 \quad (2.2)$$

We would call this pmf the Poisson - Weighted Akash distribution (P-WAD). It can be easily verified that PAD (1.1) is a particular case of P-WAD for $\alpha=1$. The natures of P-WAD for varying values of the parameters θ and α have been explained graphically in figure 1. It is observed that pmf is decreasing as increased value of θ whereas pmf is decreasing as increased value of α.



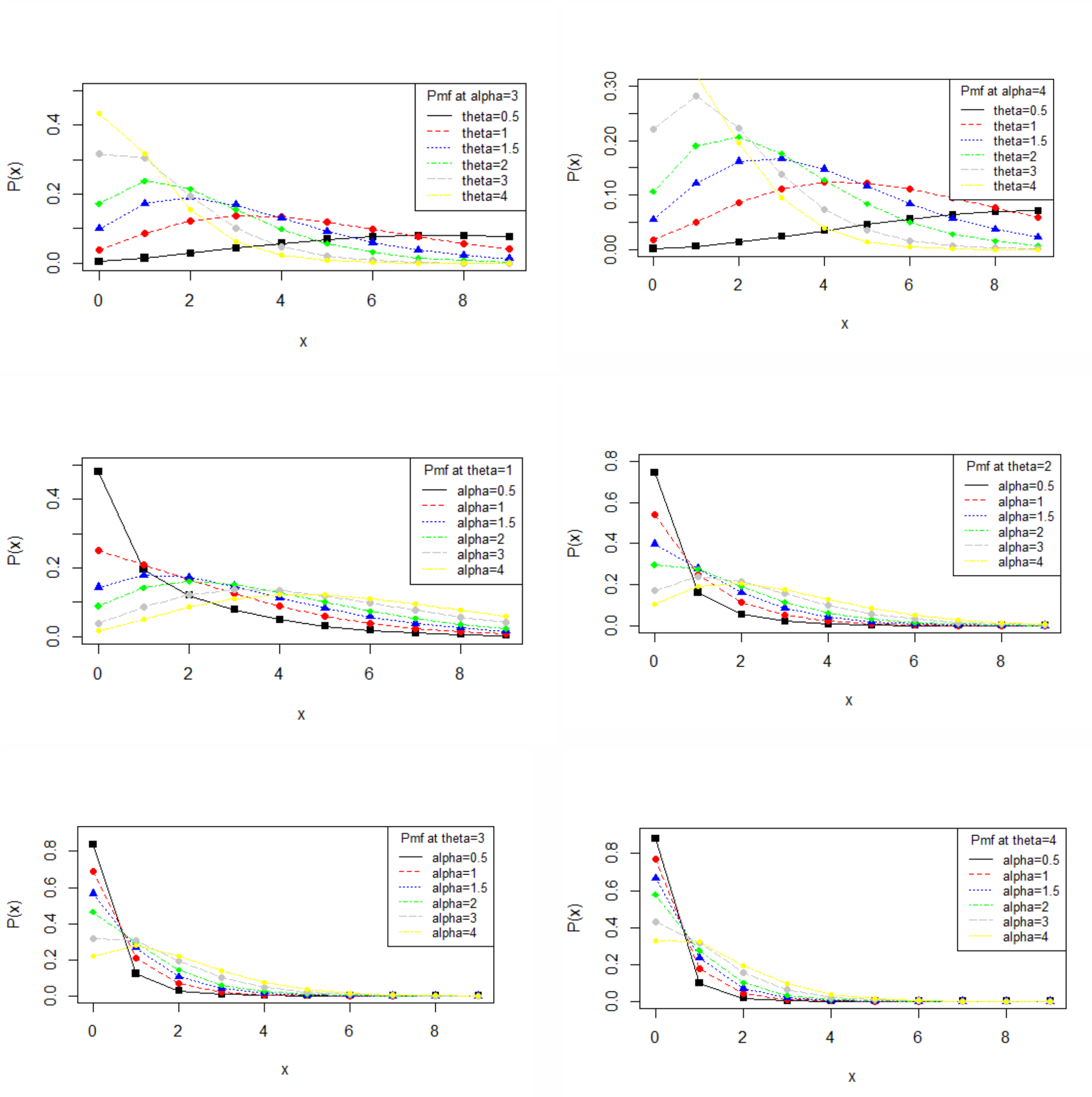


Figure 1. Probability mass function plot of P-WAD for varying values of parameters θ and α

3. Moments, skewness, kurtosis and index of dispersion

Using (2.1), the r th factorial moment about origin of the P-WAD (2.2) can be obtained as

$$\mu_{(r)}' = E\left[E\left(X^{(r)} \mid \lambda\right)\right] \text{, where } X^{(r)} = X\left(X-1\right)\left(X-2\right)...\left(X-r+1\right)$$

$$= \int_0^\infty \left[\sum_{x=1}^{\infty} x^{(r)} \frac{e^{-\lambda}\lambda^x}{\Gamma(x+1)}\right] \frac{\theta^{\alpha+2}}{\left(\theta^2+\alpha^2+\alpha\right)} \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}\left(1+\lambda^2\right)e^{-\theta\lambda} d\lambda$$

$$= \frac{\theta^{\alpha+2}}{\left(\theta^2+\alpha^2+\alpha\right)\Gamma(\alpha)} \int_0^\infty \left[\lambda^r \left\{\sum_{x=r}^{\infty} x \frac{e^{-\lambda}\lambda^{x-r}}{(x-r)!}\right\}\right] \lambda^{\alpha-1}\left(1+\lambda^2\right)e^{-\theta\lambda} d\lambda$$

Taking $x - r = y$, we get

$$\mu_{(r)}' = \frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)} \int_0^\infty \left[\lambda^r \left\{\sum_{y=0}^{\infty} \frac{e^{-\lambda}\lambda^y}{y!}\right\}\right] \lambda^{\alpha-1}(1+\lambda^2)e^{-\theta\lambda}d\lambda$$

$$= \frac{\theta^{\alpha+2}}{(\theta^2+\alpha^2+\alpha)\Gamma(\alpha)} \int_0^\infty \lambda^{\alpha+r-1}(1+\lambda^2)e^{-\theta\lambda}d\lambda$$

$$= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)} \frac{\theta^2+(\alpha+r)(\alpha+r+1)}{\theta^r(\theta^2+\alpha^2+\alpha)} \quad ; r=1,2,3,.... \tag{3.1}$$

Taking $r=1,2,3,$ and 4 in (3.1), the first four factorial moments about origin of P-WAD (2.2) can be obtained

$$\mu_{(1)}' = \frac{\alpha\{\theta^2+(\alpha+1)(\alpha+2)\}}{\theta(\theta^2+\alpha^2+\alpha)}$$

$$\mu_{(2)}' = \frac{\alpha(\alpha+1)\{\theta^2+(\alpha+2)(\alpha+3)\}}{\theta^2(\theta^2+\alpha^2+\alpha)}$$

$$\mu_{(3)}' = \frac{\alpha(\alpha+1)(\alpha+2)\{\theta^2+(\alpha+3)(\alpha+4)\}}{\theta^3(\theta^2+\alpha^2+\alpha)}$$

$$\mu_{(4)}' = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)\{\theta^2+(\alpha+4)(\alpha+5)\}}{\theta^4(\theta^2+\alpha^2+\alpha)}.$$

Now using the relationship between factorial moments about origin and the moments about origin, the first four moments about origin of P-WAD (2.2) can be obtained as

$$\mu_1' = \frac{\alpha(\theta^2+\alpha^2+3\alpha+2)}{\theta(\theta^2+\alpha^2+\alpha)}$$

$$\mu_2' = \frac{\alpha\{\theta^3+(\alpha+1)\theta^2+(\alpha^2+3\alpha+2)\theta+(\alpha^3+6\alpha^2+11\alpha+6)\}}{\theta^2(\theta^2+\alpha^2+\alpha)}$$

$$\mu_3' = \frac{\alpha\left\{\begin{array}{l}\theta^4+3(\alpha+1)\theta^3+2(\alpha^2+3\alpha+2)\theta^2+3(\alpha^3+6\alpha^2+11\alpha+6)\theta \\ +(\alpha^4+10\alpha^3+35\alpha^2+50\alpha+24)\end{array}\right\}}{\theta^3(\theta^2+\alpha^2+\alpha)}$$

$$\mu_4' = \frac{\alpha\left\{\begin{array}{l}\theta^5+7(\alpha+1)\theta^4+7(\alpha^2+3\alpha+2)\theta^3+8(\alpha^3+6\alpha^2+11\alpha+6)\theta^2 \\ +6(\alpha^4+10\alpha^3+35\alpha^2+50\alpha+24)\theta+(\alpha^5+15\alpha^4+85\alpha^3+225\alpha^2+274\alpha+120)\end{array}\right\}}{\theta^4(\theta^2+\alpha^2+\alpha)}$$

Now, using the relationship $\mu_r = E(Y-\mu_1')^r = \sum_{k=0}^{r}\binom{r}{k}\mu_k'(-\mu_1')^{r-k}$ between central moments and the moments about origin, the central moments of the P-WAD (2.2) can be obtained as

$$\mu_2 = \frac{\alpha\left\{\begin{array}{l}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta \\ +\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{array}\right\}}{\theta^2\left(\theta^2+\alpha^2+\alpha\right)^2}$$

$$\mu_3 = \frac{\alpha\left\{\begin{array}{l}\theta^8+3\theta^7+\left(3\alpha^2+5\alpha+4\right)\theta^6+\left(9\alpha^2+27\alpha+18\right)\theta^5+\left(3\alpha^4+10\alpha^3+17\alpha^2+34\alpha+24\right)\theta^4 \\ +\left(9\alpha^4+42\alpha^3+57\alpha^2+24\alpha\right)\theta^3+\left(\alpha^6+5\alpha^5+15\alpha^4+31\alpha^3+32\alpha^2+12\alpha\right)\theta^2 \\ +\left(3\alpha^6+15\alpha^5+27\alpha^4+21\alpha^3+6\alpha^2\right)\theta+\left(2\alpha^6+10\alpha^5+18\alpha^4+14\alpha^3+4\alpha^2\right)\end{array}\right\}}{\theta^3\left(\theta^2+\alpha^2+\alpha\right)^3}$$

$$\mu_4 = \frac{\alpha\left\{\begin{array}{l}\theta^{11}+\left(3\alpha+7\right)\theta^{10}+\left(4\alpha^2+12\alpha+14\right)\theta^9+\left(12\alpha^3+52\alpha^2+85\alpha+48\right)\theta^8 \\ +\left(6\alpha^4+42\alpha^3+138\alpha^2+246\alpha+144\right)\theta^7+\left(18\alpha^5+114\alpha^4+296\alpha^3+370\alpha^2+290\alpha+120\right)\theta^6 \\ +\left(4\alpha^6+54\alpha^5+294\alpha^4+730\alpha^3+774\alpha^2+288\alpha\right)\theta^5 \\ +\left(12\alpha^7+100\alpha^6+340\alpha^5+642\alpha^4+772\alpha^3+550\alpha^2+168\alpha\right)\theta^4 \\ +\left(\alpha^8+30\alpha^7+230\alpha^6+712\alpha^5+1041\alpha^4+722\alpha^3+192\alpha^2\right)\theta^3 \\ +\left(3\alpha^9+31\alpha^8+132\alpha^7+350\alpha^6+643\alpha^5+747\alpha^4+470\alpha^3+120\alpha^2\right)\theta^2 \\ +\left(6\alpha^9+60\alpha^8+228\alpha^7+432\alpha^6+438\alpha^5+228\alpha^4+48\alpha^3\right)\theta \\ +\left(3\alpha^9+30\alpha^8+114\alpha^7+216\alpha^6+219\alpha^5+114\alpha^4+24\alpha^3\right)\end{array}\right\}}{\theta^4\left(\theta^2+\alpha^2+\alpha\right)^4}$$

The coefficient of variation $(C.V)$, coefficient of Skewness $\left(\sqrt{\beta_1}\right)$, coefficient of Kurtosis (β_2) and index of dispersion (γ) of the P-WAD (2.2) are thus obtained as

$$C.V = \frac{\sigma}{\mu_1'} = \frac{\sqrt{\left\{\begin{array}{l}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta \\ +\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{array}\right\}}}{\sqrt{\alpha}\left(\theta^2+\alpha^2+3\alpha+2\right)}$$

$$\sqrt{\beta_1} = \frac{\mu_3}{\mu_2^{3/2}} = \frac{\left\{\begin{array}{l}\theta^8+3\theta^7+\left(3\alpha^2+5\alpha+4\right)\theta^6+\left(9\alpha^2+27\alpha+18\right)\theta^5+\left(3\alpha^4+10\alpha^3+17\alpha^2+34\alpha+24\right)\theta^4 \\ +\left(9\alpha^4+42\alpha^3+57\alpha^2+24\alpha\right)\theta^3+\left(\alpha^6+5\alpha^5+15\alpha^4+31\alpha^3+32\alpha^2+12\alpha\right)\theta^2 \\ +\left(3\alpha^6+15\alpha^5+27\alpha^4+21\alpha^3+6\alpha^2\right)\theta+\left(2\alpha^6+10\alpha^5+18\alpha^4+14\alpha^3+4\alpha^2\right)\end{array}\right\}}{\sqrt{\alpha}\left\{\begin{array}{l}\theta^5+\theta^4+2\left(\alpha^2+2\alpha+1\right)\theta^3+2\left(\alpha^2+4\alpha+3\right)\theta^2+\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\theta \\ +\left(\alpha^4+4\alpha^3+5\alpha^2+2\alpha\right)\end{array}\right\}^{3/2}}$$

$$\beta_2=\frac{\mu_4}{\mu_2^{\,2}}=\frac{\left\{\begin{array}{l}\theta^{11}+(3\alpha+7)\theta^{10}+(4\alpha^2+12\alpha+14)\theta^9+(12\alpha^3+52\alpha^2+85\alpha+48)\theta^8\\+(6\alpha^4+42\alpha^3+138\alpha^2+246\alpha+144)\theta^7+(18\alpha^5+114\alpha^4+296\alpha^3+370\alpha^2+290\alpha+120)\theta^6\\+(4\alpha^6+54\alpha^5+294\alpha^4+730\alpha^3+774\alpha^2+288\alpha)\theta^5\\+(12\alpha^7+100\alpha^6+340\alpha^5+642\alpha^4+772\alpha^3+550\alpha^2+168\alpha)\theta^4\\+(\alpha^8+30\alpha^7+230\alpha^6+712\alpha^5+1041\alpha^4+722\alpha^3+192\alpha^2)\theta^3\\+(3\alpha^9+31\alpha^8+132\alpha^7+350\alpha^6+643\alpha^5+747\alpha^4+470\alpha^3+120\alpha^2)\theta^2\\+(6\alpha^9+60\alpha^8+228\alpha^7+432\alpha^6+438\alpha^5+228\alpha^4+48\alpha^3)\theta\\+(3\alpha^9+30\alpha^8+114\alpha^7+216\alpha^6+219\alpha^5+114\alpha^4+24\alpha^3)\end{array}\right\}}{\alpha\left\{\begin{array}{l}\theta^5+\theta^4+2(\alpha^2+2\alpha+1)\theta^3+2(\alpha^2+4\alpha+3)\theta^2+(\alpha^4+4\alpha^3+5\alpha^2+2\alpha)\theta\\+(\alpha^4+4\alpha^3+5\alpha^2+2\alpha)\end{array}\right\}^2}$$

$$\gamma=\frac{\sigma^2}{\mu_1'}=\frac{\left\{\begin{array}{l}\theta^5+\theta^4+2(\alpha^2+2\alpha+1)\theta^3+2(\alpha^2+4\alpha+3)\theta^2+(\alpha^4+4\alpha^3+5\alpha^2+2\alpha)\theta\\+(\alpha^4+4\alpha^3+5\alpha^2+2\alpha)\end{array}\right\}}{\theta(\theta^2+\alpha^2+\alpha)(\theta^2+\alpha^2+3\alpha+2)}.$$

Behaviors of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of P-WAD for varying values of parameters θ and α have been shown graphically in figure 2.

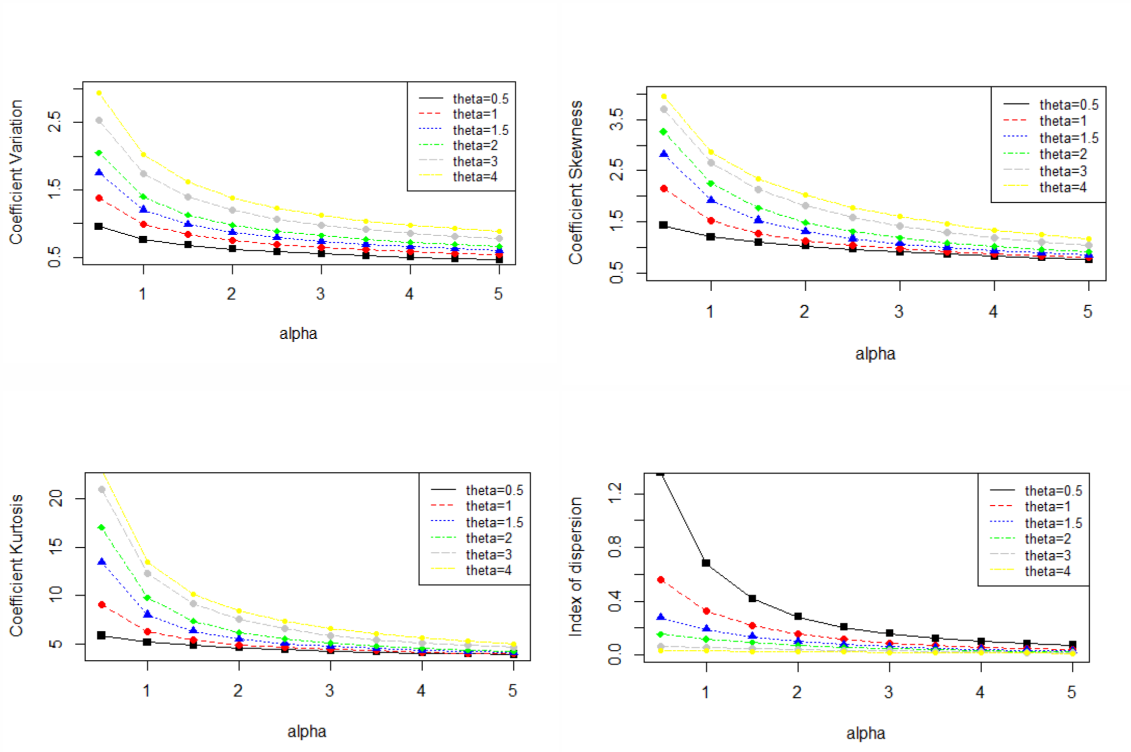


Figure 2. Behaviors of coefficient of variation, coefficient of skewness, coefficient of kurtosis and index of dispersion of P-WAD for varying values of parameters θ and α

4. Maximum likelihood estimation

Let $(x_1, x_2, ..., x_n)$ be a random sample of size n from the P-WAD (2.2) and let f_x be the observed frequency in the sample corresponding to $X = x \ (x = 1, 2, 3, ..., k)$ such that $\sum_{x=1}^{k} f_x = n$, where k is the largest observed value having non-zero frequency. The log likelihood function of P-WAD (2.2) can be given by

$$\log L = n\left[(\alpha+2)\log\theta - \log\left(\theta^2+\alpha^2+\alpha\right)\right] + \sum_{x=1}^{k} f_x\left[\log\Gamma(x+\alpha) - \log\Gamma(\alpha) - \log(x+1)\right]$$
$$-\sum_{x=1}^{k} f_x(x+\alpha+2)\log(\theta+1) + \sum_{x=1}^{k} f_x \log\left[x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]$$

The maximum likelihood estimates $(\hat{\theta}, \hat{\alpha})$ of (θ, α) of P-WAD (2.2) is the solutions of the following log likelihood equations

$$\frac{\partial \log L}{\partial\theta} = \frac{n(\alpha+2)}{\theta} - \frac{2n\theta}{\theta^2+\alpha^2+\alpha} - \sum_{x=1}^{k}\frac{(x+\alpha+2)f_x}{\theta+1}$$
$$+\sum_{x=1}^{k}\frac{2(\theta+1)f_x}{x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)} = 0$$

$$\frac{\partial \log L}{\partial\alpha} = n\log\theta - \frac{n(2\alpha+1)}{\theta^2+\alpha^2+\alpha} + \sum_{x=1}^{k} f_x\left[\psi(x+\alpha)-\psi(\alpha)\right] - \sum_{x=1}^{k} f_x\log(\theta+1)$$
$$+\sum_{x=1}^{k}\frac{(2x+2\alpha+1)f_x}{x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)} = 0$$

where $\bar{x}$ is the sample mean and $\psi(x+\alpha) = \frac{d}{d\alpha}\log\Gamma(x+\alpha)$ and $\psi(\alpha) = \frac{d}{d\alpha}\log\Gamma(\alpha)$ are digamma functions. These two log likelihood equations do not seem to be solved directly. However, the Fisher's scoring method can be applied to solve these equations. We have

$$\frac{\partial^2 \log L}{\partial\theta^2} = -\frac{n(\alpha+2)}{\theta^2} - \frac{2n\left(\alpha^2-\theta^2+\alpha\right)}{\left(\theta^2+\alpha^2+\alpha\right)^2} + \sum_{x=1}^{k}\frac{(x+\alpha+2)f_x}{(\theta+1)^2}$$
$$+\sum_{x=1}^{k}\frac{2\left\{x^2+(2\alpha+1)x+\left(\alpha^2-\theta^2-2\theta+\alpha-1\right)\right\}f_x}{\left[x^2+(2\alpha+1)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \alpha^2} = \frac{-n\left\{2\left(\theta^2+\alpha^2+\alpha\right)-\left(2\alpha+1\right)^2\right\}}{\left(\theta^2+\alpha^2+\alpha\right)^2} + \sum_{x=1}^{k}\left[f_x \psi'\left(x+\alpha\right)-\psi'\left(\alpha\right)\right]$$
$$+\sum_{x=1}^{k} \frac{\left[2\left\{x^2+\left(2\alpha+1\right)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right\}-\left(2x+2\alpha+1\right)^2\right]f_x}{\left[x^2+\left(2\alpha+1\right)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha} = \frac{n}{\theta} + \frac{2n\theta\left(2\alpha+1\right)}{\left(\theta^2+\alpha^2+\alpha\right)^2} - \sum_{x=1}^{k}\frac{f_x}{\theta+1} - \sum_{x=1}^{k}\frac{2\left(\theta+1\right)^2 f_x}{\left[x^2+\left(2\alpha+1\right)x+\left(\theta^2+\alpha^2+2\theta+\alpha+1\right)\right]^2} = \frac{\partial^2 \log L}{\partial \alpha \partial \theta}$$

where $\psi'\left(x+\alpha\right)=\frac{d}{d\alpha}\psi\left(x+\alpha\right)$ and $\psi'\left(\alpha+1\right)=\frac{d}{d\alpha}\psi\left(\alpha+1\right)$ are trigamma functions.

The maximum likelihood estimates $\left(\hat{\theta},\hat{\alpha}\right)$ of $\left(\theta,\alpha\right)$ of P-WAD (2.2) is the solution of the following equations

$$\begin{bmatrix} \frac{\partial^2 \log L}{\partial \theta^2} & \frac{\partial^2 \log L}{\partial \theta \partial \alpha} \\ \frac{\partial^2 \log L}{\partial \alpha \partial \theta} & \frac{\partial^2 \log L}{\partial \alpha^2} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}} \begin{bmatrix} \hat{\theta}-\theta_0 \\ \hat{\alpha}-\alpha_0 \end{bmatrix} = \begin{bmatrix} \frac{\partial \log L}{\partial \theta} \\ \frac{\partial \log L}{\partial \alpha} \end{bmatrix}_{\substack{\hat{\theta}=\theta_0 \\ \hat{\alpha}=\alpha_0}}$$

where θ_0 and α_0 are the initial values of θ and α respectively. These equations are solved iteratively till sufficiently close values of $\hat{\theta}$ and $\hat{\alpha}$ are obtained.

5. Applications

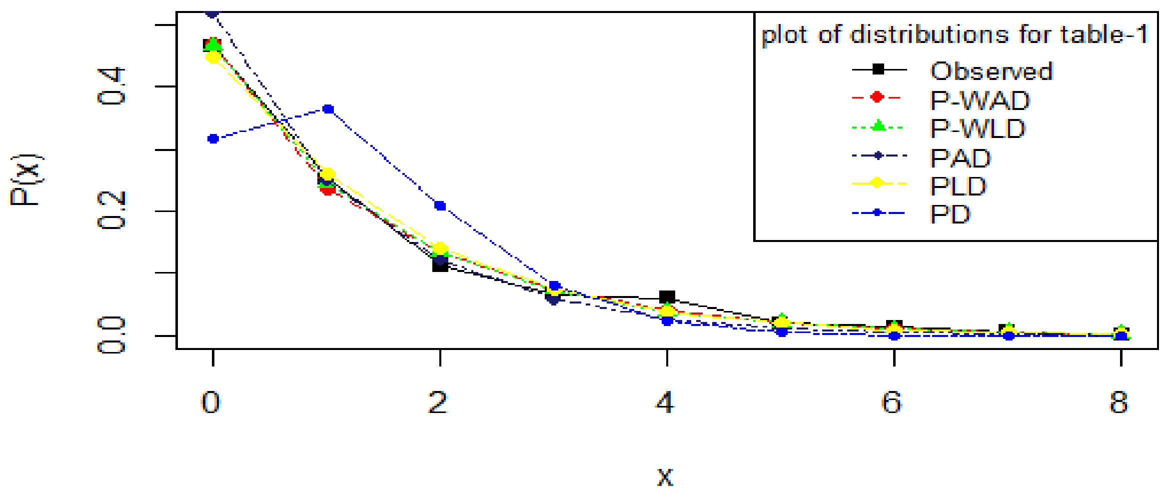
In this section the applications of the P-WAD has been discussed with two count datasets from biological sciences. The dataset in table 1 is the data regarding the number of European red mites on apple leaves, available in Bliss (1953). The dataset in 2 is the frequencies of the observed number of days that experienced X thunderstorm events at Cape Kennedy, Florida for the 11-year period of record in the month of June and July, January 1957 to December 1967 and are available in Falls *et al* (1971) and Carter (2001). The goodness of fit of P-WAD has been compared with the goodness of fit given by Poisson distribution (PD), PLD, PAD, and P-WLD. Note that the estimates of the parameters are based on maximum likelihood estimates for all the considered distributions. Based on the values of chi-square$\left(\chi^2\right)$, $-2\log L$ and AIC (Akaike Information criterion), it is obvious that P-WAD is competing well with the considered distributions and gives better fit. Note that AIC has been calculated using the formula $AIC=-2\log L+2k$, where k is the number of parameters involved in the distribution.

Table 1. Observed and Expected number of European red mites on Apple leaves, available in Bliss (1953)

Number of Red mites per leaf	Observed frequency	Expected frequency				
		PD	PLD	PAD	P-WLD	P-WAD
0	70	47.6	67.2	78.0	69.8	70.6
1	38	54.6	38.9	37.3	36.8	35.6
2	17	31.3	21.2	18.3	20.1	20.0
3	10	11.9	11.1	8.8	10.9	11.1
4	9	3.4	5.7	4.1	5.8	6.0
5	3	0.8	2.8	1.8	3.0	3.2
6	2	0.2	1.4	0.8	1.6	1.6
7	1	0.1	0.9	0.3	0.8	0.8
8	0	0.1	0.8	0.6	1.2	1.1
Total	150	150.0	150.0	150.0	150.0	150.0
ML estimates		$\hat{\theta}=1.14666$	$\hat{\theta}=1.26010$	$\hat{\theta}=1.89341$	$\hat{\theta}=1.09141$ $\hat{\alpha}=0.82194$	$\hat{\theta}=1.4585$ $\hat{\alpha}=0.8360$
Standard Errors		0.08743	0.11390	0.13240	0.26231 0.25230	0.12627 0.06936
χ^2		26.50	2.49	8.29	2.41	2.29
d.f		2	4	3	3	3
p-value		0.0000	0.5595	0.04038	0.4917	0.5144
$-2\log L$		485.61	445.02	447.02	425.35	439.41
AIC		487.61	447.02	449.02	429.35	443.41

Table 2. Frequencies of the observed number of days that experienced X thunderstorm events at Cape Kennedy, Florida for the 11-year period of record in the month of June, January 1957 to December 1967

X	Observed frequency	Expected frequency				
		PD	PLD	PAD	P-WLD	P-WAD
0	187	155.6	185.3	190.7	185.1	187.6
1	77	116.9	83.4	79.7	83.7	80.5
2	40	43.9	35.9	34.4	36.0	35.4
3	17	11.0	15.0	14.7	15.0	15.4
4	6	2.0	6.1	6.1	6.1	6.5
5	2	0.3	2.5	2.5	2.4	2.7
6	1	0.3	1.8	1.9	1.7	1.9
Total	330	330.0	330.0	330.0	330.0	330.0
ML estimate		$\hat{\theta}=0.75148$	$\hat{\theta}=1.80427$	$\hat{\theta}=2.17976$	$\hat{\theta}=1.82188$ $\hat{\alpha}=1.01237$	$\hat{\theta}=2.15124$ $\hat{\alpha}=1.01198$
Standard Errors $\hat{\theta}$ $\hat{\alpha}$		0.04772	0.12573	0.10781	0.41748 0.28219	0.13789 0.05056
χ^2		31.6	1.43	1.64	1.41	1.31
d.f		2	3	3	2	2
p-value		0.0000	0.6985	0.6503	0.4941	0.5194
$-2\log L$		824.50	788.88	840.66	874.20	788.84
AIC		826.50	790.88	842.66	878.20	788.73



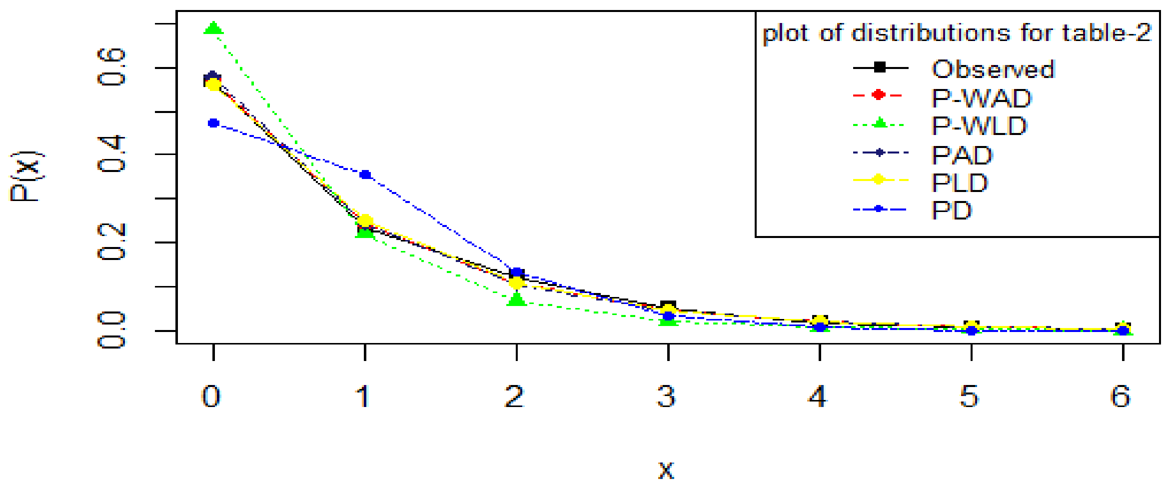


Figure 3. Fitted probability plots for distributions

6. Concluding remarks

A Poisson-Weighted Akash distribution which includes Poisson-Akash distribution has been proposed. Its moments and moments based statistical constants have been derived and studied. Some statistical properties have been discussed. Maximum likelihood estimation has been discussed for estimating parameters of the distribution. Finally, applications of the proposed distribution have been explained through some count datasets and the goodness of fit has been compared with other discrete two parameter and one parameter distributions and it was found satisfactory over P-WLD, PAD, PLD, and PD on considered data sets.

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TESTS FOR EQUALITY OF VARIANCES BETWEEN TWO SAMPLES WHICH CONTAIN BOTH PAIRED OBSERVATIONS AND INDEPENDENT OBSERVATIONS

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Abstract

Tests for equality of variances between two samples which contain both paired observations and independent observations are explored using simulation. New solutions which make use of all of the available data are put forward. These new approaches are compared against standard approaches that discard either the paired observations or the independent observations. The approaches are assessed under equal variances and unequal variances, for two samples taken from the same distribution. The results show that the newly proposed solutions offer Type I error robust alternatives for the comparison of variances, when both samples are taken from the same distribution.

Key words: Brown-Forsythe test; Equal variances; Partially overlapping samples; Pitman-Morgan test; Simulation; Robustness

1. Introduction

An equality of variances test is often performed as a preliminary test to inform the most appropriate statistical test for a comparison of means (Mirtagioğlu *et al.* 2017). The pitfalls of this process are well documented (Zimmerman, 2004; Zimmerman and Zumbo, 2009; Rasch *et al.*, 2011; Rochon *et al.*, 2012). This paper considers tests for equality of variances where it is the equality of variances that is of importance in their own right. Examples include a comparison of two treatments that have a similar mean efficacy, or a comparison of products in quality control, or a comparison of variances in human populations. Tests for equal variances have wide ranging applications including areas in archaeology, environmental science, business and medical research (Gastwirth *et al.*, 2009).

Numerous tests for the comparisons of variances for two independent samples have been documented (Conover, *et al.,* 1981). The Pitman-Morgan test is widely regarded as the optimum test of equal variances with two paired samples under normality (Mudholkar *et al.*, 2003). However, situations may arise where there are two samples which contain both independent observations and paired observations (Derrick *et al.,* 2015). For example, when some experimental data in a paired samples design is missing due to an error or accident.

This paper is concerned with the direct comparison of variances between two samples, which contain both paired observations and independent observations. For simplicity, these scenarios are referred to as partially overlapping samples (Martinez-Camblor *et al.,* 2013; Derrick *et al.*, 2017). The conditions of Missing Completely at Random (MCAR) are assumed.

In the two partially overlapping samples scenario, if the number of paired observations is relatively large and the number of independent observations is relatively small, a solution may be to discard independent observations and perform a test for equal variances on the paired observations. The standard F-test is not appropriate for paired samples (Kenny, 1953). For the comparison of variances for paired data, the Pitman-Morgan test can be performed (Pitman 1938; Morgan 1939). However, the Pitman-Morgan test is not robust to violations of the assumption of normality (Mudholkar *et al.*, 2003; Grambsch, 2015). For heavy tailed distributions the Type I error rate of the Pitman-Morgan test is larger than nominal Type I error rate (McCulloch, 1987; Wilcox, 2015).

Alternatively, if the number of independent observations is relatively large and the number of paired observations is relatively small, a solution may be to discard paired observations and perform one of numerous established tests for the comparison of variances with independent observations.

When the normality assumption is met, the standard F-test is the uniformly most powerful test for two independent samples. However, the standard F-test is not robust to deviations from normality (Marozzi, 2011).

Levene (1960) proposed that for two independent groups, the differences between the absolute deviations from the group means could be used to assess equality of variances. In the two sample case, this test is equivalent to Student's t-test applied to absolute deviations from the group means. This version of Levene's test, fails to control the Type I error rate when the population distribution is skewed (Carroll and Schneider, 1985; Nordstokke and Zumbo, 2007).

Brown and Forsythe (1974) proposed alternatives to Levene's test when data are not normally distributed. These alternatives use deviations from the median or trimmed mean. These variations are also often referred to as "Levene's test" (Carroll and Schneider, 1985; Gastwirth *et al.*, 2009). For the avoidance of doubt, in this paper the convention followed is that assessing equality of variances using deviations from the mean is referred to as Levene's test. Assessing equality of variances using deviations from the median is referred to as the Brown-Forsythe test.

Conover *et al*. (1981) explored 56 tests for equal variances for two independent groups and noted that the five tests that are Type I error robust use deviations from the median rather than deviations from the mean. Conover *et al*. (1981) found that the only test that consistently meets Bradley's (1978) liberal Type I error robustness criteria is the Brown-Forsythe test, using absolute deviations from the median. There is no uniformly robust and most powerful test applicable for all distributions and sample sizes. The general consensus is praise of the Brown-Forsythe test using deviations from the median (Carroll and Schneider, 1985; Nordstokke and Zumbo, 2007; Mirtagioğlu *et al.,* 2017). However, it should be noted that this test can be conservative with small sample sizes (Loh, 1987; Lim and Loh, 1995). The use of absolute deviations rather than squared deviations better maintains Type I error robustness (Cody and Smith, 1997).

Performing a test using either only the independent observations or only the paired observations may result in loss of power. The discarding of data is particularly problematic if the overall total sample size is small. In addition, if the assumption of MCAR is not reasonable, the discarding of data is likely to cause bias.

Bhoj (1979, 1984) and Ekbohm (1981, 1982) debated methods using all of the available data for testing the equality of variances in scenarios that they refer to as "incomplete data". In this debate the authors do not recognise that a combination of independent observations and paired observations may occur by design and not only by accident. Bhoj (1979) and Ekbohm (1981, 1982) independently considered a weighted combination of existing independent sum of squares techniques to create a new test statistic. Other solutions such as ignoring the pairing and performing the F-test on all of the available data were considered by Ekbohm (1982). Bhoj (1984) concluded that his test statistic is the most powerful if the correlation is negative or small. Otherwise, performing the F-test on all of the available data is more powerful than the solutions put forward by either of the authors (Ekbolm, 1982; Bhoj 1984). The simulations performed by these authors were on a relatively small scale, with only 1,000 replicates at each point in their design space. No solution was comprehensively agreed upon for all scenarios, and this is likely to contribute to them not being well established. Furthermore the non-robustness of the Pitman-Morgan test has a detrimental impact on their weighted tests. A solution that uses all available data without a complex weighting structure, or the discarding of valuable information about the pairing, may therefore be advantageous.

For the comparison of means when both independent observations and paired observations are present, partially overlapping samples t-tests are given by Derrick, *et al.* (2017). These solutions are generalised forms of the t-test and are Type I error robust under normality. These solutions are also robust in the comparison of two ordinal samples where the scale represents interval data (Derrick and White, 2018).

We propose that as an alternative test of equal variances when there is a combination of paired observations and independent observations, the partially overlapping samples t-test can be performed, using deviations from the group medians, as outlined below.

Let X_{ji} denote the *i*-th observation in group *j* for *j* = {Sample1, Sample 2}, and $\tilde{X}_j$ denote the sample median, so that $Y_{ji} = \left| X_{ji} - \tilde{X}_j \right|$, then

$$T_{var1} = \frac{\bar{Y}_1 - \bar{Y}_2}{S_{p\_y}\sqrt{\frac{1}{n_1} + \frac{1}{n_2} - 2r\left(\frac{n_c}{n_1 n_2}\right)}} \text{ and } S_{p\_y} = \sqrt{\frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{(n_1 - 1) + (n_2 - 1)}}$$

The test statistic T_{var1} is referenced against the t-distribution with degrees of freedom:

$$v_1 = (n_c - 1) + \left(\frac{n_a + n_b + n_c - 1}{n_a + n_b + 2n_c}\right)(n_a + n_b).$$

where $n_a =$ number of unpaired observations exclusive to Sample 1, $n_b =$ number of unpaired observations exclusive to Sample 2, $n_c =$ number of pairs, $n_j =$ total number of observations in Sample *j*, $S_j^2 =$ variance of Sample *j* based on the Y_{ji} observations.

For the comparison of variances, Loh (1987) suggested adapting the unequal variances t-test using deviations from the medians. For the comparison of means, Student's t-test is sensitive to deviations from the equal variances assumption (Ruxton, 2006; Derrick, Toher and White, 2016). As a result of this Derrick *et al*. (2017) additionally proposed the partially overlapping samples t-test for unequal variances. We propose that the partially overlapping samples test statistic unconstrained to equal variances can be similarly modified to provide a test for equality of variances so that:

$$T_{var2} = \frac{\bar{Y}_1 - \bar{Y}_2}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2} - 2r\left(\frac{S_1 S_2 n_c}{n_1 n_2}\right)}}$$

The test statistic T_{var2} is referenced against the t-distribution with degrees of freedom:

$$v_2 = (n_c - 1) + \left(\frac{\gamma - n_c + 1}{n_a + n_b + 2n_c}\right)(n_a + n_b) \text{ where } \gamma = \frac{\left(\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}\right)^2}{\frac{\left(S_1^2/n_1\right)^2}{n_1 - 1} + \frac{\left(S_2^2/n_2\right)^2}{n_2 - 1}}$$

Methodology for assessing the Type I error rate of these proposals is given in Section 2, with an example application given in Section 3.

2. Methodology

For two samples containing both independent observations and paired observations, approaches for the comparison of variances are assessed using simulation. The approaches considered are the Brown-Forsythe test, the Pitman-Morgan test, and the proposed T_{var1} and T_{var2}. Type I error robustness is assessed using Bradley's (1978) liberal robustness criteria. Power is assessed for test statistics that do not violate Bradley's liberal criteria.

Within the simulation design, the sizes of n_a, n_b, n_c are {5, 10, 30, 50}. The correlation coefficients ρ are {0.00, 0.25, 0.50, 0.75}. Simulations for each possible parameter combination of n_a, n_b, n_c, ρ are performed in a factorial design. Standard Normal deviates are calculated using the Box-Muller (1958) transformation. For the n_c observations, correlated Standard Normal deviates are obtained as per Kenney and Keeping (1951)

In Section 4.1, the comparison of variances is performed for normally distributed data. Under the null hypothesis, X_1 ~ N(0,1) and X_2 ~ N(0,1). Under the alternative hypothesis, the observations in Sample 2 are multiplied by two, thus X_1 ~ N(0,1) and X_2 ~ N(0,4).

In Section 4.2, the comparison of variances is performed for skewed distributions. Under the null hypothesis, Normal deviates are first generated as above, and then the exponential of each value is calculated. Under the alternative hypothesis this process is repeated, and each of the observations in Sample 2 are multiplied by two to create unequal variances.

For each parameter combination, the data generating process is repeated 10,000 times, and each of the statistical tests to be evaluated is performed on each replicate. Under the null hypothesis, the proportion of the replicates where the null hypothesis is rejected represents the Type I error rate. Under the alternative hypothesis, the proportion of the replicates where the null hypothesis is rejected, represents the power of the test, assuming Type I error rates can be reasonably compared. The simulations and tests are performed in R, at the 5% significance level, two-sided.

The simulation design allows that the conditions of MCAR can be assumed.

3. Example

In the assessment of an undergraduate university module, two lecturers share the marking of 32 student submissions. As part of the marking regulations, at random six of the submissions are independently assessed by both lecturers. The remaining submissions are randomly split between the two lecturers, ensuring that both have an equal number to assess. Thus Lecturer 1 has one sample comprising of six paired observations and 13 independent observations. Likewise, Lecturer 2 has a sample of equal size. The samples are partially overlapping by design, thus MCAR can be reasonably assumed.

There is concern that the lecturers do not allocate marks at the top end and the bottom end of the marking scale in the same way. Tests for equal variances are performed on the independent observations (Table 1), the paired observations (Table 2), and all observations.

Table 1. Marks awarded to the 26 students randomly allocated to the lecturers.

Lecturer 1	55	56	58	60	60	60	61	61	62	62	64	65	67
Lecturer 2	40	50	51	60	60	60	60	60	61	66	69	72	82

Table 2. Marks awarded by each lecturer for the six students that are marked by both.

Student	A	B	C	D	E	F
Lecturer 1	54	55	60	63	65	70
Lecturer 2	50	56	60	61	67	73

The Brown-Forsythe test is performed on the data in Table 1 using the R package "lawstat" (Gastwirth *et al.*, 2015). This shows no evidence to reject the null hypothesis of equal variances (t = -1.9673, v = 24, p = 0.061).

The Pitman-Morgan test is performed on the data in Table 2 using the R package "PairedData" (Champely, 2013). This shows no evidence to reject the null hypothesis of equal variances (t = -2.352, v = 4, p = 0.078).

In order to perform the tests for equal variances using all of the available data, for each submission marked my Lecturer 1 the absolute deviation from the median mark given by Lecturer 1 is calculated. Similarly, the absolute deviations for Lecturer 2 are calculated.

The partially overlapping samples t-test is performed on the absolute deviations using the R package "Partiallyoverlapping" (Derrick, 2017). The null hypothesis of equal variances is rejected at the 5% significance level for both the equal variances assumed variant (t_{var1} = -2.324, v_1 = 26.211, p = 0.028) and the equal variances not assumed variant (t_{var2} = -2.183, v_2 = 17.488, p = 0.043). It would appear that Lecturer 2 is making greater use of the full range of potential marks relative to Lecturer 1.

3.1. Comparison of variances for two samples from the Normal distribution

Type I error rates and power are summarised for each of; the Brown-Forsythe test, BF, the Pitman-Morgan test, PM, and the partially overlapping samples tests, T_{var1} and T_{var2}. Each of the test statistics are assessed under the null hypothesis where $X_1 \sim$ N (0,1) and $X_2 \sim$ N (0,1). The Type I error robustness for each of the parameter combinations within the simulation design are summarised in Figure 1.

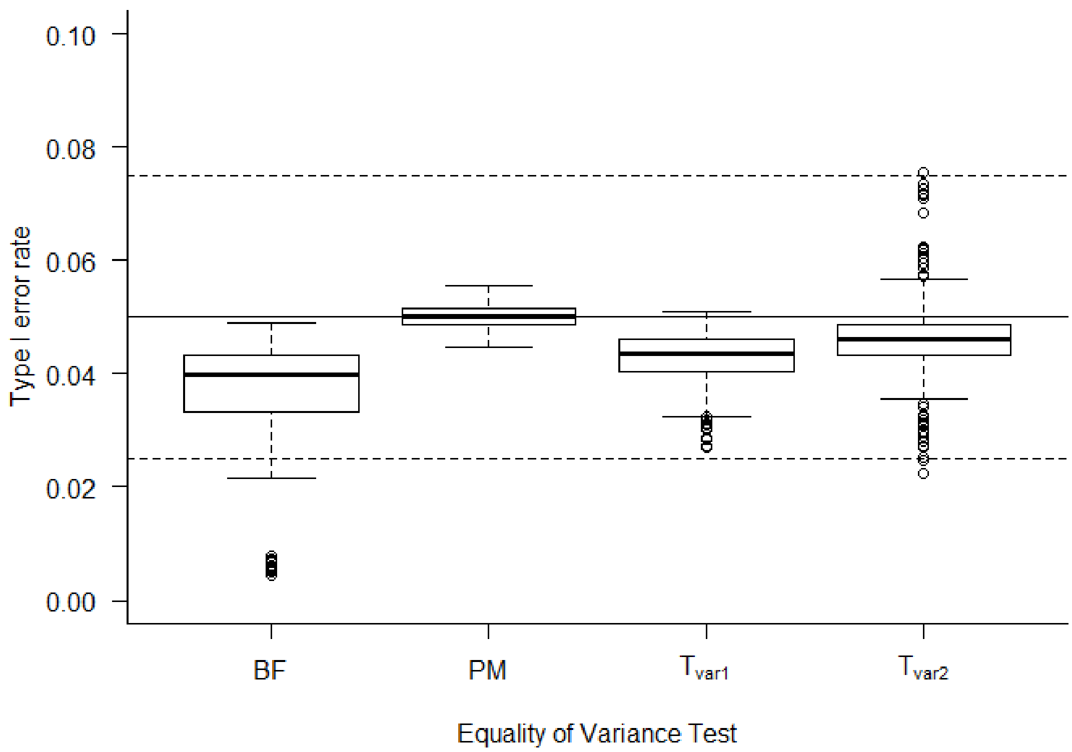

Figure 1. Type I error robustness for each parameter combination, assessed against Bradley's liberal criteria, samples from Standard Normal distribution

Figure 1 shows that the Pitman-Morgan test and the proposed test statistics are Type I error robust throughout the simulation design, with T_{var1} being more conservative than T_{var2}. For the smallest sample sizes within the design, the Brown-Forsyth test is very conservative.

Relative power comparisons for each of the test statistics are assessed where X_1 ~ N (0,1) and X_2~ N (0,4). The power averaged across the simulation design for increasing ρ is given in Figure 2.

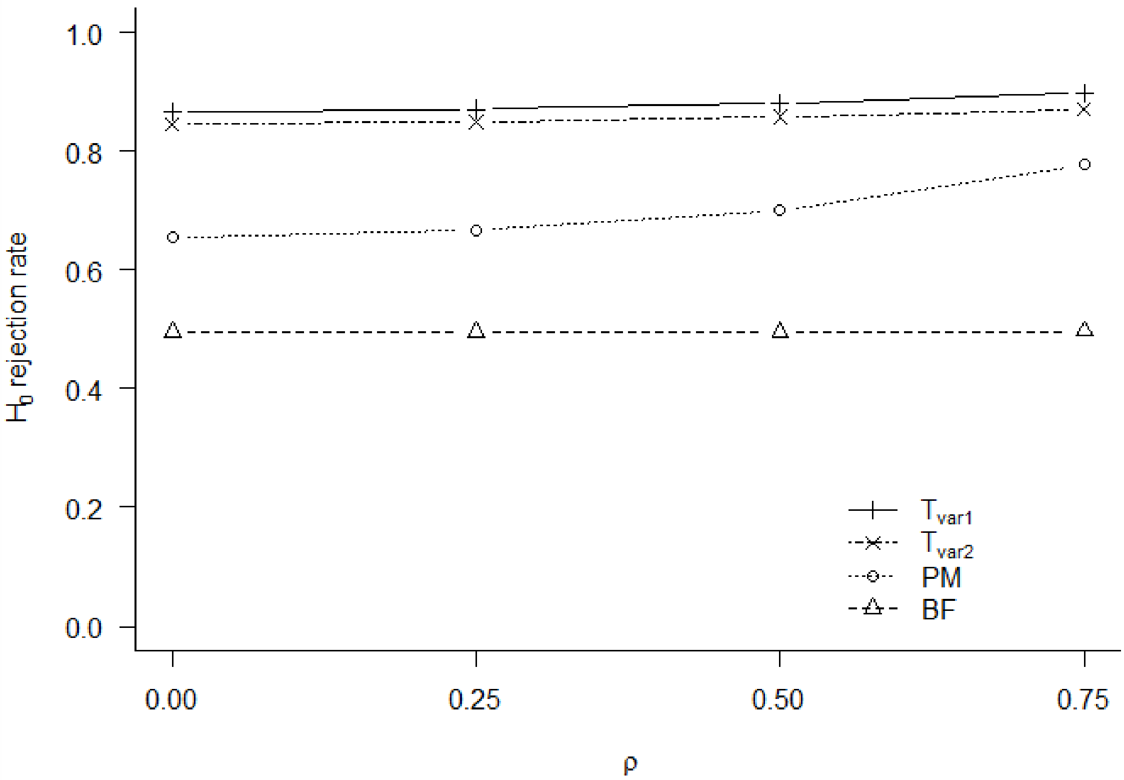

Figure 2. Relative power, averaged across the simulation design for increasing ρ, samples from Normal distributions.

Figure 2 shows that the proposed test statistics T_{var1} and T_{var2} perform similarly to each other under normality, and they have superior power qualities to the standard tests which discard data.

3.2. Comparison of variances for two samples from skewed distributions

Each of the test statistics are assessed when both samples are taken from skewed but identical distributions. The Type I error robustness for each of the parameter combinations within the simulation design are summarised in Figure 3.

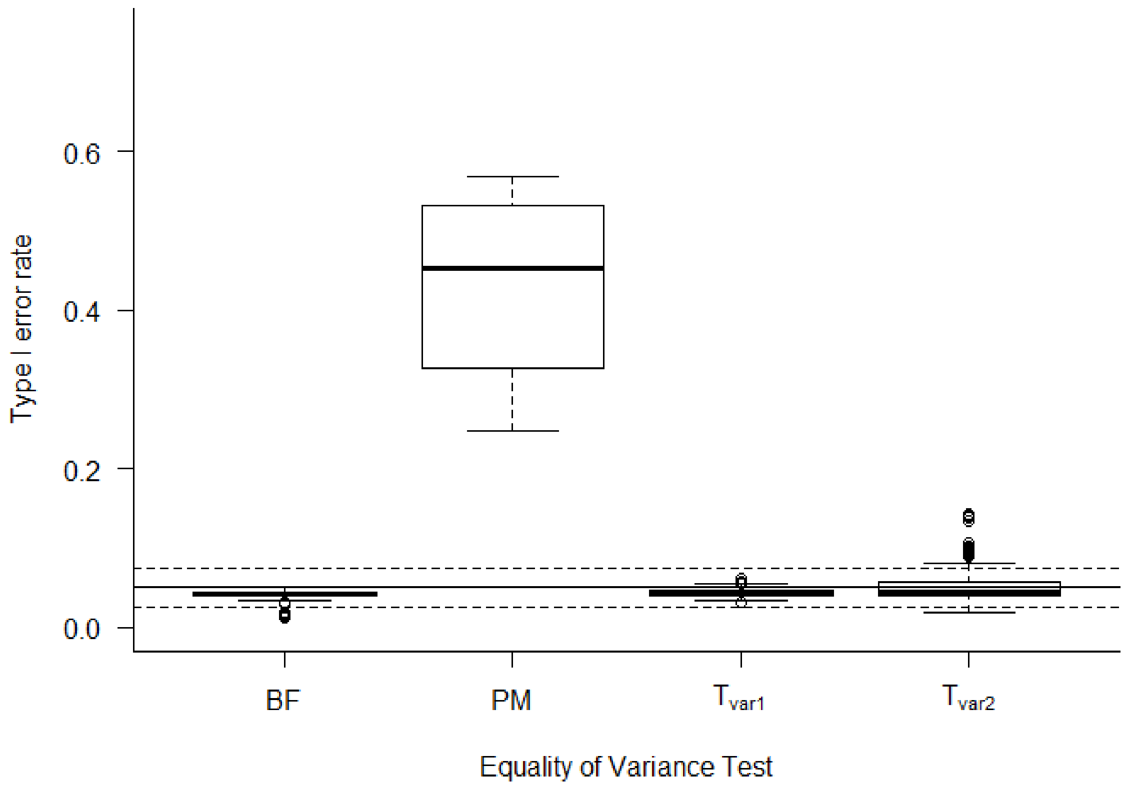

Figure 3. Type I error robustness for each parameter combination, assessed against Bradley's liberal criteria, samples from skewed distribution.

Figure 3 shows that the Pitman-Morgan test is not Type I error robust when the samples are taken from identical heavy tailed distributions. This supports the findings by McCulloch (1987) and Wilcox (2015). In addition it can be seen that T_{var2} does not fully maintain Type I error robustness. Further investigation shows that T_{var2} is liberal when one of the samples is more dominant in terms of size, and when there is a large imbalance between the number of independent observations and the number of pairs.

Relative power comparisons for each of the test statistics are assessed where the samples are taken from different skewed distributions. Due to the poor Type I error robustness of the Pitman-Morgan test and T_{var2}, this comparison is done only for the Brown-Forsythe test and T_{var1}. The power averaged across the simulation design for increasing ρ is given in Figure 4.

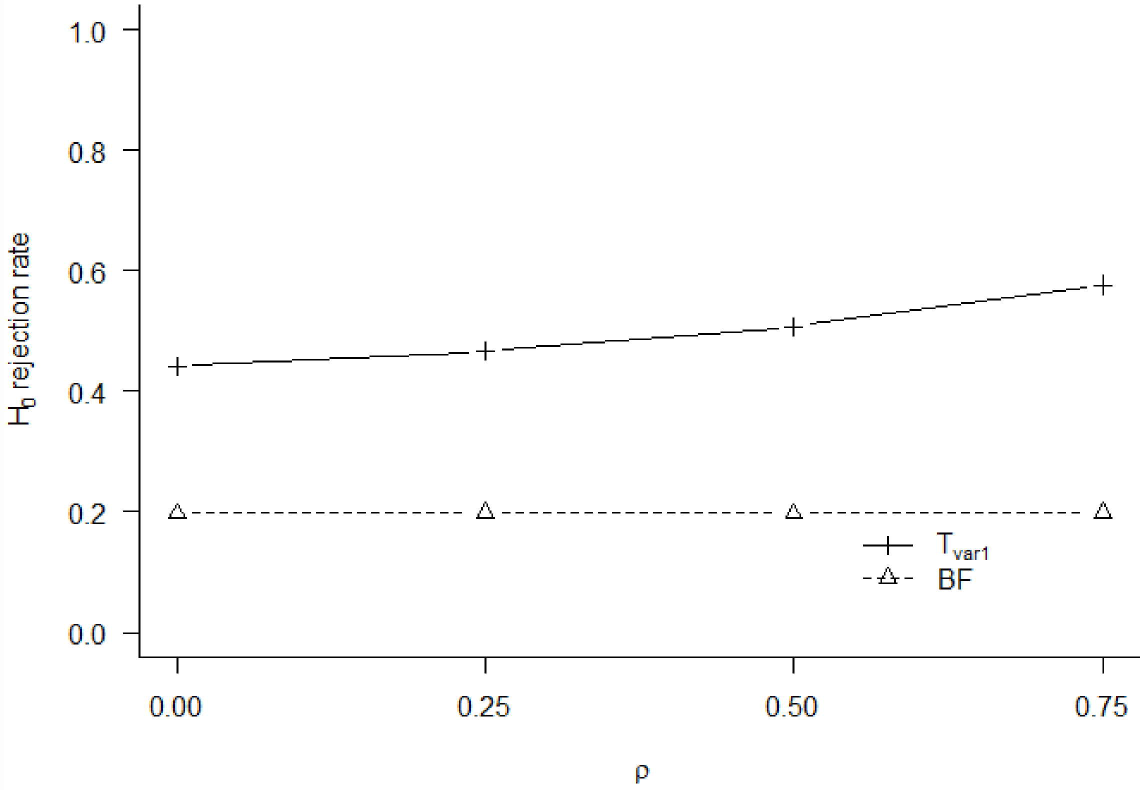

Figure 4. Relative power, averaged across the simulation design for increasing ρ, samples from skewed distributions.

Figure 4 shows that the proposed solution, T_{var1}, is more powerful than the Brown-Forsythe test. A comparison of Figure 4 against Figure 2 also indicates that both the Brown-Forsythe test and the newly proposed test, T_{var1}, are less powerful when samples are taken from a heavy-tailed distribution.

4. Conclusion

A common research question in psychology, education, medical sciences, business and manufacturing, is whether or not the variances are equal (Gastwirth, Gel and Miao, 2009).

There has been little research into techniques for the comparison of variances for samples that contain both independent observations and paired observations. Standard solutions that involve discarding data are less than desirable. Two solutions that make use of the tests statistics by Derrick *et al.* (2017) are proposed in this paper. Simulations across a range of sample sizes show that these solutions are Type I error robust under normality and the assumption of MCAR. These solutions are more powerful than established solutions that discard data, namely the Pitman-Morgan test and the Brown-Forsythe test.

The equal variances form of the partially overlapping samples variances test, T_{var1}, is marginally more powerful than the unconstrained form of the test T_{var2}.

The proposed test statistic T_{var1} further maintains Type I error robustness for skewed distributions where T_{var2} does not. T_{var1} is therefore recommended as a powerful alternative to test for the equality of variances between two samples when there is a combination of paired observations and independent observations in two samples.

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NON-FORMAL EDUCATION IN ROMANIA – AN ANALYSIS IN EUROPEAN CONTEXT

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Abstract

Non-formal education is an important component in adults' education. The current study aims to analyse the status quo of non-formal education in Europe, based on most recently available data. The research will focus on Romania. Furthermore, the paper contains an analysis of the higher education area in this country and recommendations for using non-formal education in order to bring value-added to this sector.

Keywords: *non-formal education; long life learning; adults' education; higher education*

1. Introduction

Chisholm (2005) outlines that non-formal education comprises voluntary acts of structured learning that take place outside formal education. Emphasizing the importance of such an education, Kiilakoski (2015) points out that non-formal education is an essential part of adults' professional development, helping them to develop soft skills that are not sufficiently covered by formal education.

Moreover, Patrick (2010) recommends that non-formal education should be recognised as part of formal studies. European Centre for the Development of Vocational Training (2009) clearly states that even if "*validating non-formal and informal learning poses challenges to formal education in terms of the range of learning that can be validated and how this process can be integrated into the formal curriculum and its assessment*" (European Centre for the Development of Vocational Training, 2009 p.71), "*validation of non-formal and informal learning should be seen as an integral part of the national qualifications system*" (European Centre for the Development of Vocational Training,2009 p.70). Furthermore, in 2013 the European Parliament adopted *Resolution 1930 and Recommendation 2014 (2013): Young Europeans: an urgent educational challenge*, asking member states to take all necessary steps to ensure recognition and fair access to non-formal education.

Despite the importance of non-formal education and the progress made towards recognition, there are many challenges that European countries have to overcome (European Commission, 2015). Out of these, universities' reluctance to recognize non-formal education is urgent (Darnesin et al. 2014).

The current study aims to perform an analysis of the status quo of non-formal education in Europe, based on most recently available data. The research will focus on Romania, as this country made important steps for enhancing adult education (Balica, 2016). The paper is structured as follows: the first part focuses on non-formal education in Europe, in order to determine Romania's position with regard to specific indicators in this area; the second part is dedicated to various aspects of non-formal education in Romania; the last section contains an analysis of the higher education area in this country and recommendations for using non-formal education in order to bring value-added to this sector.

2. Non-formal education in Europe

Figure 1 shows the participation rate in non-formal education and training while figure 2 presents the participation rate in job-related non-formal education and training in 2007, 2011 and 2016 for several European countries, as well as for the European Union in its current composition and the Euro Area. As one can observe, Romania registers the lowest values among all countries, for both indicators, in 2016. Moreover, both indicators decreased in 2016 compared to 2011 for this country. The situation is similar for Bulgaria, Malta, Estonia, Denmark, Luxemburg, Finland, Norway and Sweden. Furthermore, for Romania both indicators have lower values compared to 2007.

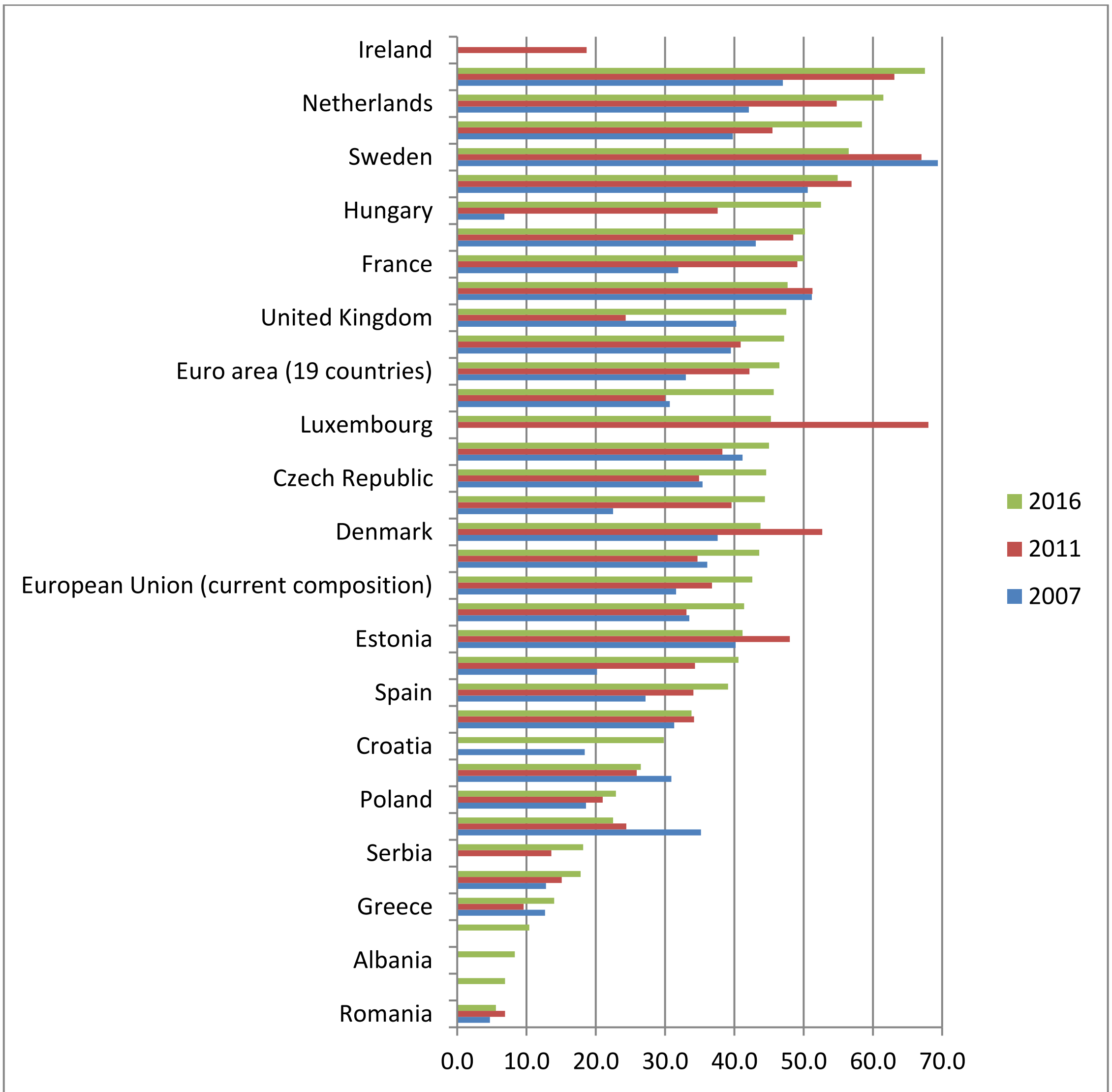

Figure 1. Participation rate in non-formal education and training
Source of data: Eurostat

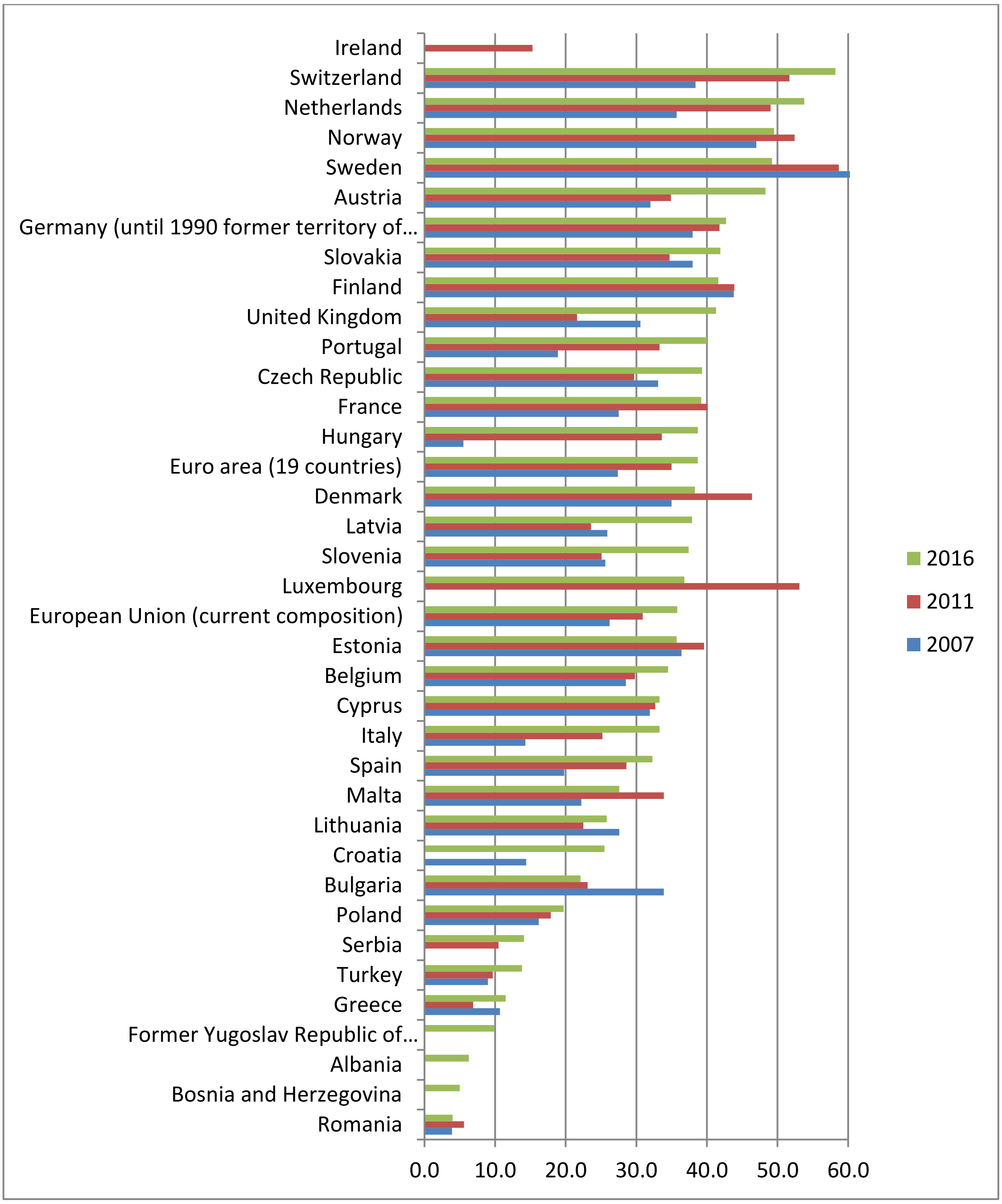

Figure 2. Participation rate in job-related non-formal education and trening
Source of data: Eurostat

Next, the analysis explores participation rates in non-formal education and training taking into account several issues identified by literature: relationship with formal education (Livingstone, 2011), gender equality (Gee, 2015), widening participation in lifelong learning for older people (Villar and Celdran, 2013). The ANOVA procedure was used in order to assess whether or not there is a significant difference in the participation rates in non-formal education by gender, age group, as well as between the participation rate in non-formal education and the participation rate in formal and non-formal education. The analysis was performed for 2007, 2011 and 2016, taking into account all European countries, for which data are available

Table 1 presents the results for the participation rate in non-formal education and training. As one can observe, there is no significant difference between this indicator and the participation rate in formal and non-formal education at European level. Also, no significant difference between genders can be observed for the participation rate in non-formal education and training. With regard to age groups, the only significant difference occurs when comparing each age group to 55 to 64 years. 47.4% of those aged 25 to 34 in the European Union participated in non-formal education in 2016. The indicator registers 46.7% for those aged 35 to 44, 43.9% for those aged 45 to 54 and only 32.3% for those aged 55 to 64.

Table 1. ANOVA single factor analysis results - P-value Participation rate in education and training

	2007	2011	2016
Formal and non-formal education and training compared to non-formal education and training	0.381645	0.423011	0.502257
Non-formal education and training - Males compared to Females	0.966904	0.905579	0.981933
Non-formal education and training - 25 to 34 years compared to 35 to 44 years	0.788014	0.950071	0.848709
Non-formal education and training - 25 to 34 years compared to 45 to 54 years	0.166973	0.376814	0.190683
Non-formal education and training - 25 to 34 years compared to 55 to 64 years	2.91E-06	2.84E-05	2.77E-05
Non-formal education and training - 35 to 44 years compared to 45 to 54 years	0.279102	0.4114	0.278691
Non-formal education and training - 35 to 44 years compared to 55 to 64 years	1.71E-05	3.73E-05	0.000104
Non-formal education and training - 45 to 54 years compared to 55 to 64 years	0.00095	0.000961	0.004271

Source: author's design

Table 2 presents the results for the participation rate in job-related non-formal education and training, while table 3 presents the results only for those activities sponsored by the employer. The only significant difference can be observed only when comparing each age group to the 55 to 64. Indeed, in 2016, 39.5% of those aged 25 to 34 participated in job-related non-formal education. The indicator registers 40.6% for those aged 35 to 44, 38.2 for those aged 45 to 54 and only 24.3% for those aged 54 to 64. Also, 34.2% of those aged 25 to 34 participated in job-related non-formal education sponsored by the employer, in 2016. This indicator registered 36.2% for those aged 35 to 44, 34.8% for those aged 45 to 54 and only 21.8% for those aged 54 to 64.

Table 2. ANOVA single factor analysis results - P-value Participation rate in job-related non-formal education and training

	2007	2011	2016
Non-formal education and training - Males compared to Females	0.393905	0.476341	0.568652
Non-formal education and training - 25 to 34 years compared to 35 to 44 years	0.985087	0.815188	0.767008
Non-formal education and training - 25 to 34 years compared to 45 to 54 years	0.347412	0.413213	0.413213
Non-formal education and training - 25 to 34 years compared to 55 to 64 years	7.49E-07	1.97E-06	1.06E-05
Non-formal education and training - 35 to 44 years compared to 45 to 54 years	0.356761	0.426542	0.275551
Non-formal education and training - 35 to 44 years compared to 55 to 64 years	1.99E-06	7.33E-07	4.88E-06
Non-formal education and training - 45 to 54 years compared to 55 to 64 years	0.000105	2.41E-05	0.000358

Source: author's design

Table 3. ANOVA single factor analysis results - P-value Participation rate in job-related and sponsored by the employer non-formal education and training

	2007	2011	2016
Non-formal education and training - Males compared to Females	0.283305	0.354112	0.429301
Non-formal education and training - 25 to 34 years compared to 35 to 44 years	0.795727	0.663182	0.544214
Non-formal education and training - 25 to 34 years compared to 45 to 54 years	0.569953	0.835279	0.793224
Non-formal education and training - 25 to 34 years compared to 55 to 64 years	1.11E-05	3.24E-05	0.000117
Non-formal education and training - 35 to 44 years compared to 45 to 54 years	0.428862	0.522817	0.391607
Non-formal education and training - 35 to 44 years compared to 55 to 64 years	1.04E-05	5.99E-06	1.35E-05
Non-formal education and training - 45 to 54 years compared to 55 to 64 years	0.000239	8.52E-05	0.000379

Source: author's design

3. Non-formal education in Romania

Participation in non-formal education in Romania is analysed based on the data provided by the National Institute of Statistics in the publication "Adults education in 2016". Based on the existing scientific literature, several specific topics were chosen for the analysis. First, as Uisalli (2017) points out, non-formal education can address various women educational needs. Second, one should note that Nayar (1979) and Combs and Ahmed (1974) concluded that non-formal education is crucial for socio-economic development of rural areas and poor regions. Third, according to Ololube and Egbezor (2012), non-formal education is a powerful tool to ensure access to basic education for adults. 5.4% of men and 5.7% of women aged 25 to 64 participated in non-formal education, resulting in an overall participation rate of 5.6%. Noticeable differences can be observed when taking into account area of residency: the participation rate for those living in urban areas was 7.2% while for those living in rural areas only 3.3%. This indicator registered 0.8% for persons aged 25 to 64 with lower than secondary education, 4.8% for those with secondary education and 13.6% for those with higher education.

Figure 3 displays participation rates at regional level. As one can observe, the lowest participation rate in non-formal education is registered for South-West Oltenia Region (3%) followed by North-East (approximately 4%). According to Poverty Mapping in Romania Making Better Policies through Better-Targeted Interventions designed by the World Bank (2014), these are the poorest regions in Romania.

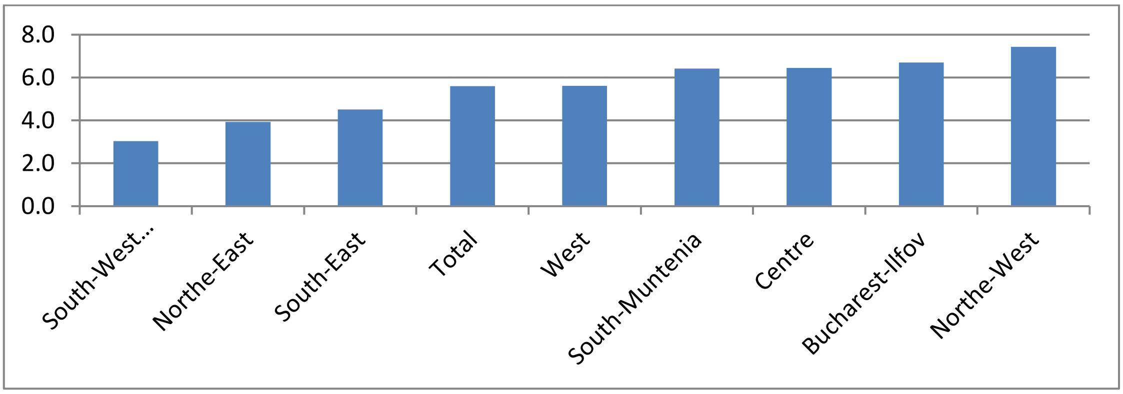

Figure 3. Persons aged 25-64 years participating in non-formal education, by region
Source of data: National Institute of Statistics of Romania

Participation rates in non-formal education for professional purposes as well as sponsored by the employer for employed persons by domain are displayed in figure 4. The lowest participation rate is registered for Agriculture, forestry and fishing (0.9%) for both indicators. One should note that, according to the latest press release of the National Institute of Statistics with regard to employment and unemployment, 22.8% of the employed persons and 19.8% of the skilled employed persons worked in this area (National Institute of Statistics, 2018).

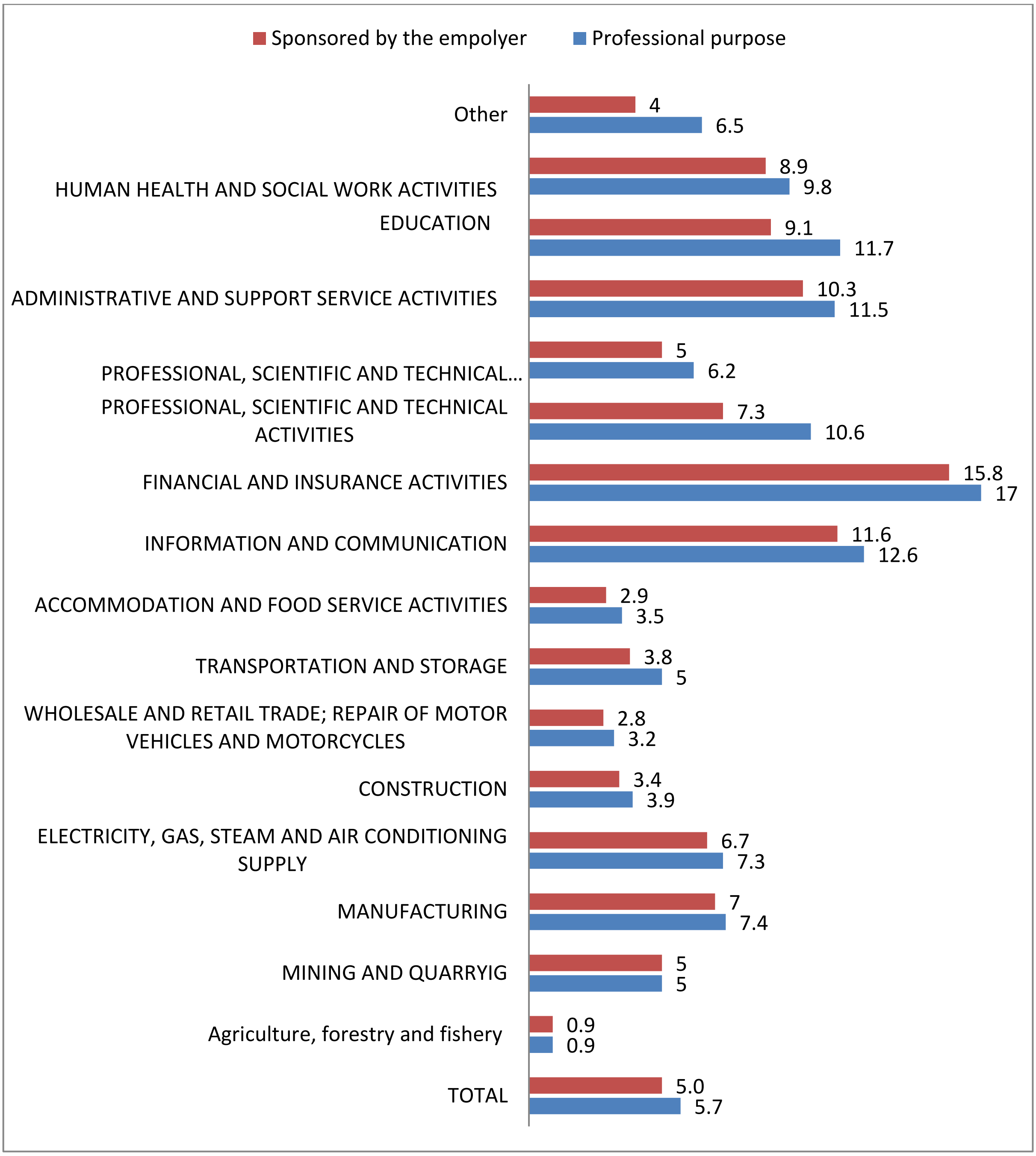

Figure 4. Employed persons participating in non-formal education by field (as percentage of employed persons in that field)

Source of data: National Institute of Statistics of Romania

Figures 5 and 6 display participation rates in non-formal education by highest educational attainment and type of activity and highest educational attainment and Internet usage respectively. Kapadia (2014) points out that training on the job is extremely important for organisations and employees. Approximately 30% of the non-formal activities consisted of training on the job. The lowest value for this indicator is registered for persons graduating lower than secondary education. Also, one should note that only one third of the participants in non-formal education used online resources.

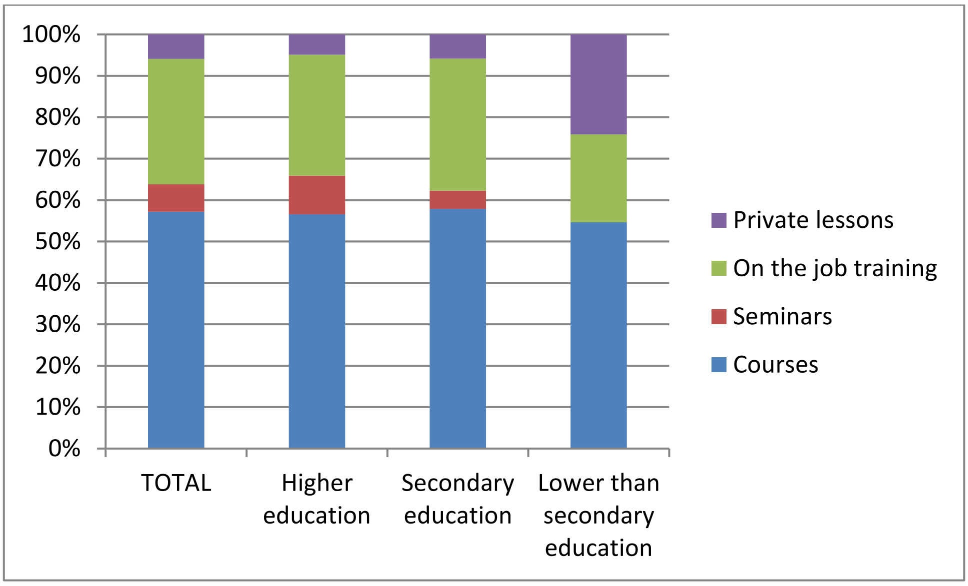

Figure 5. Participation in non-formal education by type of activity and highest educational attainment of the participant

Source of data: National Institute of Statistics of Romania

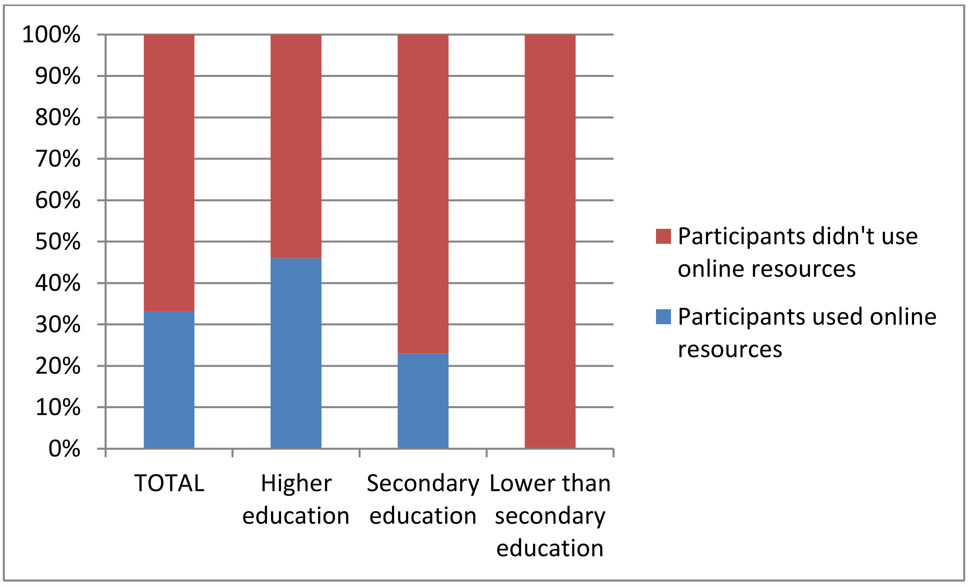

Figure 6. Participation in non-formal education highest educational attainment of the participant and online resources usage

Source of data: National Institute of Statistics of Romania

Figure 7 shows participation rates in non-formal education in the last 12 months by provider. Only 5% of non-formal activities were provided by formal education establishments. Most of the activities were provided by the employers.

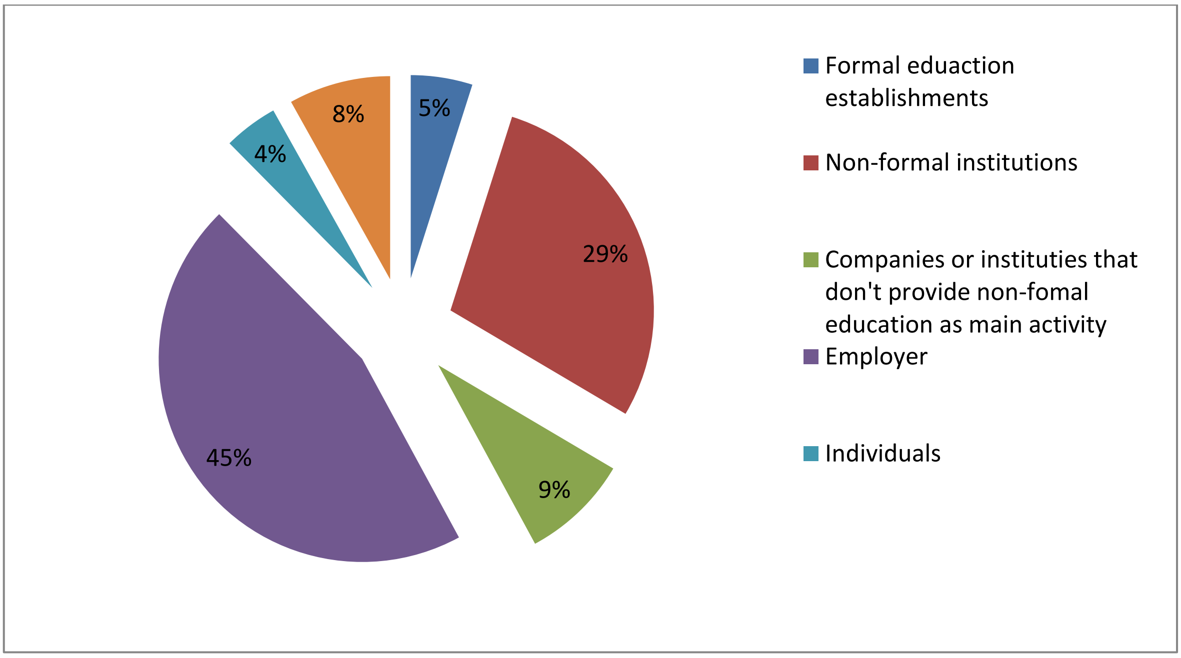

Figure 7. Participation rates in non-formal education in the last 12 months by provider
Source of data: National Institute of Statistics Romania

The outcomes obtained as a result of participation in non-formal education are displayed in figure 8. Most of those who participated in non-formal educational activities claimed that their work performance has been improved. Also, over 30% of the participants mentioned that they received new tasks. Only 9% of the participants obtained an income raise.

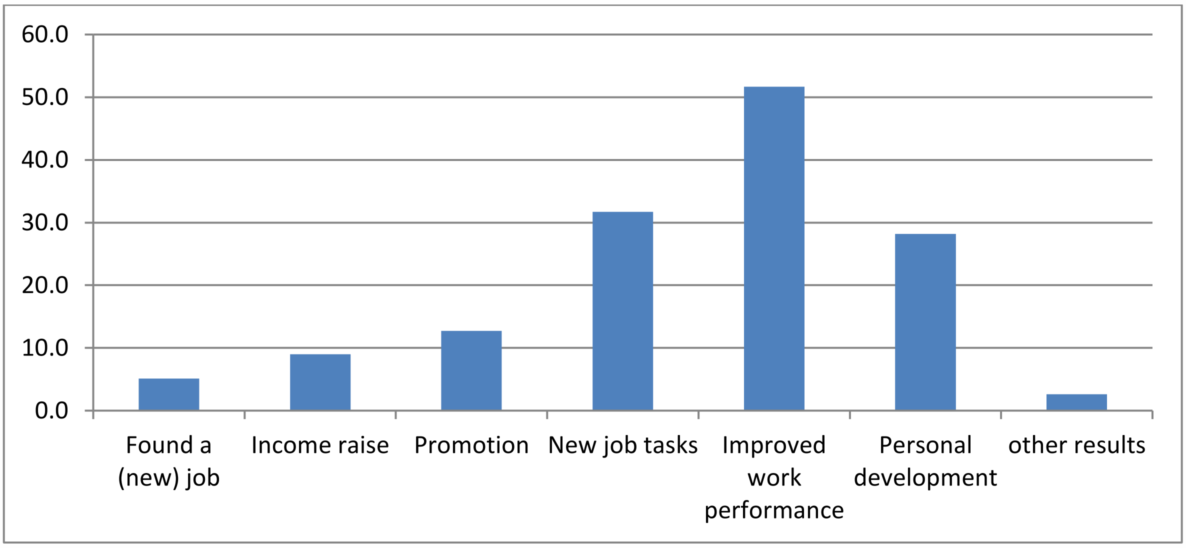

Figure 8. Outcomes obtained as a result of participation in non-formal education
Source of data: National Institute of Statistics Romania

4. Higher education in Romania – challenges and solutions from the non-formal education area

The higher education area in Romania has been facing many challenges since 1989. First, two important systematic transformations took place in the context of reformation of the entire Romanian society: private universities appeared and the number of students increased dramatically and unsustainable (Andrei *et al.,* 2010a; Andrei *et al.,* 2010b; Andrei *et al.,* 2009a). This kind of transformations occurred also in Hungary and Bulgaria (Andrei *et al.*, 2010c). One should note that the reformation process hasn't been a smooth one, as the transition period has been characterized by corruption and lack of transparency (see for example Andrei *et al.*, 2009b; Andrei *et al.,* 2009c).

Secondly, non-academic behaviour could be observed among students as well as university staff at all levels (Teodorescu and Andrei, 2009), resulting in corruption with effects on the long run (Naghdipour and Emeagwali, 2013). This further leads to slow economic development and poor quality of public services (Andrei *et al.,* 2009d, Andrei *et al.*, 2009e)

Thirdly, Romanian universities are not very attractive to foreign students (Mirica *et al.*, 2015). One reason is that research in Romanian universities doesn't have enough visibility (see for example Teodorescu and Andrei, 2014 and Andrei *et al.,* 2016).

Encouraging students to participate in non-formal education can help solve these issues. Firstly, non-formal education is more flexible and student-centred, addressing specific educational needs of youths (Luxemburg Government, 2013). Therefore, for somebody who wants to develop a specific skill, pursuing a non-formal course is much more effective than enrolling in a university and attending an entire programme. However, this works only if non-formal education is properly recognised by a society.

Secondly, *"non-formal education is a way of helping societies to be more democratic and to respect human rights"* (Parliamentary Assembly Doc. 8595 of 15 December 1999). This idea emerged from the pioneers of educational reform: Nikolaj Frederik Severin Grundtvig, who implemented the first non-formal based-learning school in Denmark in 1844 (Danish Adult Education Association, 2015) and Henry David Thoreau, who radically challenged formal education (See for example Thoreau, 1858)

Thirdly, non-formal education provided by student organisations helps increase communication among students (Mirica and Abdulamit, 2014). Moreover, with the creation of European student associations such as the European Students' Union, student networks created through student organisations can be extended internationally.

5. Conclusions

Participation in non-formal education increased at European level in 2016 compared to 2011 and 2007. Moreover, non-formal education at European level is characterised by equal opportunities for men and women. With regard to age group, persons 55 to 64 are underrepresented in this kind of education. However, different situations could be observed at country level.

Romania has the lowest participation rate in non-formal education in Europe. Analysing the situation at regional revel, one can conclude that the lowest participation rate is registered for the poorest regions. Also, taking into account the economic activity the lowest participation rate can be observed for agriculture, forestry and fishing, despite the fact that approximately 20% of the employed population works in this area. Moreover, formal educational institutions provide only 5% of the services in this area.

Formal education in Romania, especially the higher education area, has several issues. However, non-formal education can provide several solutions to address them in the context of a proper institutional framework.

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