DECISIONAL MODELS AND ASPECTS FOR OPTIMAL MANAGE OF SOME PRODUCTION PROCESSES

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Abstract: *Decisions about supplied goods depend of used technologies, of possibilities to acquire necessary factors, of quality of products, of demand level etc. Usually, firms cannot get a 100% level of qualitative goods, dividing them in some groups or categories. We research three generalizations of the classical model, where expected profit depends of quantity of good of first quality, of prices for each category, of demand level. We propose to solve described models using method of projection of generalized gradient. There are present some experimental results for different demand behavior.*

Key words: models; decisions; generalized gradient; stochastic

There is a follow linear model that expresses hypothetic value of maximal revenue of an industrial enterprise:

$$V(y) = \sum_{j=1}^{n} v_j y_j \to \max_y \qquad (1)$$

subject to:

$$\sum_{j=1}^{n} a_{ij} y_j \le b_i, i = \overline{1,m} \qquad (2)$$

$$y_j \ge 0, j = \overline{1,n} \qquad (3)$$

Significance of used notations:

v_j - price per unit of good *j*, $j = \overline{1,n}$;

y_j - quantity of good *j* – level of that will be determinate

$V(y)$ - total revenue, that firm will have obtain if will sell all amounts of produces $y_1, y_2, ..., y_n$;

b_i - available quantity of resource *i*, $i=\overline{1,m}$;

a_{ij} - technological coefficient that represents necessity of resource *i* to create a unit of product *j*;

Nearly all linear models inclusive model (1)-(3) can be solved using an universal method SIMPLEX. The model (1)-(3) represents adequate that situation when all quantity of production is sold. Therefore is reasonable to develop the shown model, which will describe different situations from an activity of production firm. Next, is present three generalizations of described model [1].

Generalization 1

Decision about quantity of each category of goods that follow to produce depends certainly of demand level on market for respective product. In such situations model (1)-(3) can take under consideration the demand for each good or service. Let us now admit that demand is represented by vector $Y=(Y_1,Y_2,...,Y_n)$. While firm makes a quantity of goods greater than demand then, it accounts additional expenses that are determinated by so-called "phenomena of overproduction". These expenses will be included in the next model. The vector $p=(p_1,p_2,...,p_n)$ define the losses per unit that system of production accounts for each sort of good of overproduce ($y>Y$).

Taking into consideration mentioned conditions, the total net revenue is represented as follows:

$$V(y;Y)=\sum_{j=1}^{n}\left[v_j\cdot\min\{y_j;Y_j\}-p_j\cdot\max\{0;y_j-Y_j\}\right] \tag{4}$$

Defining $\varphi_j(y_j;Y_j)=v_j\cdot\min\{y_j;Y_j\}-p_j\cdot\max\{0;y_j-Y_j\}$ that represents net revenue obtained from realization of good *j* and is a nondifferentiable function in relation with deciding factor y_j, then, evident:

$$\varphi_j(y_j;Y_j)=\begin{cases}v_j\cdot y_j, & for \quad y_j\le Y_j\\(v_j-p_j)Y_j-p_jy_j, & for \quad y_j>Y_j\end{cases}$$

Thus, taking in consideration new data, result the next:

$$V(y;Y)=\sum_{j=1}^{n}\varphi_j(y_j;Y_j)\to\max_y$$

subject to: $$\sum_{j=1}^{n}a_{ij}y_j\le b_i, i=\overline{1,m} \qquad y_j\ge 0, j=\overline{1,n}$$

Generalization 2

Every manager takes such decisions that will guarantee the lowest level of rejects (defective articles/goods) in total amount of manufactured goods. However, in spite of this, is unrealizable to get the 100% of qualitative goods and this situation can be analyzed just as a theoretical one in the almost cases. From this reason are justifiable the following improvement of the model obtained at the first generalization.

Further, it is assumed that $k_j \cdot 100\% \ (0 < k_j \le 1)$ from produced good y_j represents competitive production, the rest, $\alpha_j y_j \ (\alpha_j = 1 - k_j)$ represents defective goods and cannot be sale. In this case, net revenue is represented as follows:

$$V(y;Y) = \sum_{j=1}^{n} \left[v_j \cdot \min\{k_j y_j; Y_j\} - p_j \cdot \max\{y_j - Y_j; \alpha_j y_j\} \right]$$

and function respective $\varphi_j(y_j;Y_j) = \begin{cases} v_j \cdot Y_j - p_j(y_j - Y_j) & if \quad Y_j \le k_j y_j \\ v_j k_j y_j - p_j \alpha_j y_j & if \quad Y_j > k_j y_j \end{cases}$

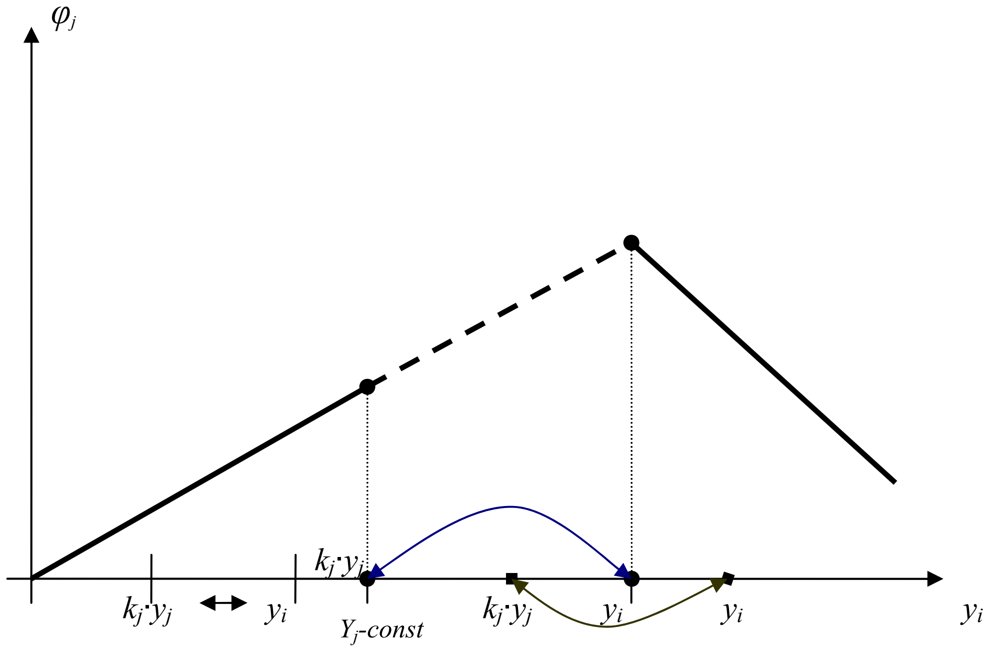

Figure 1. Dependence of net revenue in relation with product *j* , for fixed demand Y_j and coefficient of competitiveness $0<k_j<1$

Generalization 3

There is a similar situation as in previous stage, but where $k_j y_j$ represents production of "superior" quality (A), with price per unit $v_j = v_j^A$, and $\alpha_j y_j$ - production of "lower" (B) price per unit equal with v_j^B and $0 \le v_j^B < v_j^A$. Concurrently we admit that good *j* of the "second" quality can be sold only if were sold all quantity of product of quality A. Supplement at revenue will be:

a) 0 , if $Y_j \le k_j y_j$

b) $v_j^B (Y_j - k_j y_j)$, if $k_j y_j \le Y_j \le y_j$

c) $v_j^B (y_j - k_j y_j)$, if $Y_j \ge y_j$

Reunify these three cases we obtain value of addition in form:

$$\min\left\{ \max\{0; v_j^B (Y_j - k_j y_j)\}; v_j^B \alpha_j y_j \right\}$$

In addition, function $\varphi_j(y_j;Y_j)$ obtains the following aspect:

$$\varphi_j(y_j;Y_j) = v_j^A \cdot \min\{k_j y_j;Y_j\} + v_j^B \cdot \min\left\{ \max\{0;Y_j - k_j y_j\};\alpha_j y_j \right\} - p_j \cdot \max\{y_j - Y_j;0\}$$

Besides modification of objective function will be changed constraints of non-negativity of variables, that more adequately reflect behavior of an economic system:

$$0 \le y_j^{\min} \le y_j \le y_j^{\max} \quad j = \overline{1,n}$$

Necessity of such conditions appears in situations when firm cannot (constraints of different nature) exceed volumes of some kinds of goods.

Finally, we obtained nest model [2]:

$$V(y;Y) = \sum_{j=1}^{n} \varphi_j(y_j;Y_j) \to \max_y \tag{5}$$

where

$$\varphi_j(y_j;Y_j) = v_j^A \cdot \min\{k_j y_j;Y_j\} + v_j^B \cdot \min\left\{ \max\{0;Y_j - k_j y_j\};\alpha_j y_j \right\} - p_j \cdot \max\{y_j - Y_j;0\} \tag{6}$$

Subject to:

$$\sum_{j=1}^{n} a_{ij} y_j \le b_i, i = \overline{1,m} \tag{7}$$

$$0 \le y_j^{\min} \le y_j \le y_j^{\max} \quad j = \overline{1,n} \tag{8}$$

Evident, respective model is non-linear (objective function is non-linear) and cannot be solved using traditional methods. In this situation we can use the method of generalized gradient [4].

Analyzing relations between demand vector *(Y)*, total produced quantity *(y)* and manufactured amount of quality A ($k \cdot y$) we obtain 3 situations that determine 3 correspondence forms of function $\varphi_j(y_j,Y_j)$:

a) $Y_j \le k_j y_j$ - demand do not exceed volume of good *j* of quality A (competitive) in this case function φ_j has form: $\varphi_j(y_j,Y_j) = v_j^A \cdot Y_j - p_j \cdot (y_j - Y_j)$;

b) $k_j y_j \le Y_j \le y_j$ - quantity of unsatisfied demand, evident is equal with $Y_j - k_j y_j$, in this situation φ_j is represented as:

$$\varphi_j(y_j,Y_j) = v_j^A k_j y_j + v_j^B \cdot (Y_j - k_j y_j) - p_j \cdot (y_j - Y_j);$$

c) $y_j \le Y_j$ then $\varphi_j(y_j,Y_j) = v_j^A \cdot k_j \cdot y_j + v_j^B \cdot \alpha_j \cdot y_j$.

Fixing demand quantity Y_i (figure 1) in a point and varying y_i in certain limits, we obtain graphical image of function $\varphi_j(y_j,Y_j)$ in dependence of output y_i (for 3 situations as described above).

Researching price of good of quantity *B* (v_j^B) and its contribution in value of net income we observe that in situation **a**) this contribution represents 0 monetary units, because sold amount of quality *B* the same is zero. In situation **c**) all quantity of good is sold and total income increases due to products of quality *B* with $v_j^B \cdot \alpha_j \cdot y_j$ monetary units. The most interesting is situation ***b***) here we can see three moments related to contribution of goods of quality B in total income and these three cases depends of price v_j^B :

1. $v_j^B > 0$ but inessential, on graphic this situation corresponding to *case I* and optimal volume of supply is $y_j = Y_j / k_j$;
2. $v_j^B > 0$ and of such nature that $v_j^A \cdot Y_j - p_j \cdot (y_j - Y_j) = v_j^A k_j y_j + v_j^B \cdot (Y_j - k_j y_j) - p_j \cdot (y_j - Y_j)$ that is accordingly maximal income and is equal for every y_j that satisfies condition $Y_j \le y_j \le \frac{Y_j}{k_j}$. In figure 1 respective situation corresponds to *case II* ;
3. $v_j^B > 0$ for values of v_j^B when contribution of quality B is *considerable*, then y_j optimal supply will be equal with demand Y_j and respective maximal income $\varphi_j^{\max}$ is $v_j^A k_j Y_j + v_j^B \cdot (Y_j - k_j Y_j)$, *case III* from figure 2.

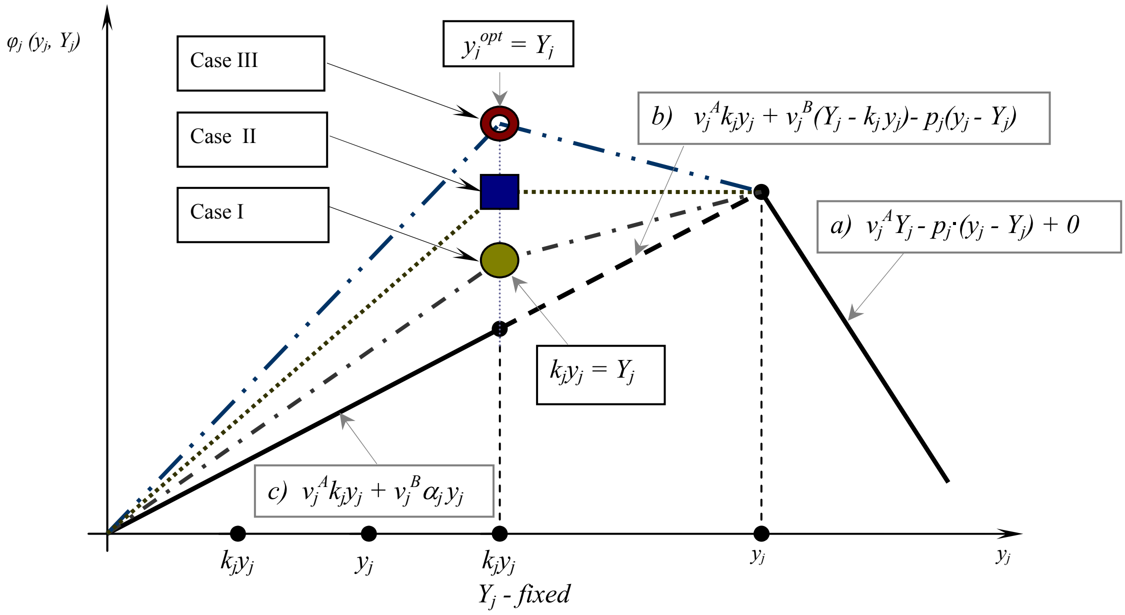

Figure 2. Graphical illustration of net income in dependence of relations between demand, supply and values of v_i^A și v_i^B, p_i, k_i

Further, will make an evaluation of price v_j^B , in function of values that were presented in those 3 cases from situation ***b***). Reasoning from equality $v_j^A \cdot Y_j - p_j \cdot (y_j - Y_j) = v_j^A k_j y_j + v_j^B \cdot (Y_j - k_j y_j) - p_j \cdot (y_j - Y_j)$ or from that angular coefficient of function $\varphi_j(y_j, Y_j) = v_j^A k_j y_j + v_j^B \cdot (Y_j - k_j y_j) - p_j \cdot (y_j - Y_j)$ must be zero while function is constant, obtain following:

$$v_j^A k_j - v_j^B k_j - p_j = 0 \Leftrightarrow v_j^B k_j = v_j^A k_j - p_j \Leftrightarrow v_j^B = \frac{v_j^A k_j - p_j}{k_j} = v_j^A - \frac{p_j}{k_j}$$

Generalizing those related above we get following conclusions:

1. if $0 \le v_j^B < v_j^A - \frac{p_j}{k_j}$ then y_j optimal is $y_j^{opt} = \frac{Y_j}{k_j}$

2. while $v_j^B = v_j^A - \frac{p_j}{k_j}$ then y_j optimal is $Y_j \le y_j^{opt} \le \frac{Y_j}{k_j}$

3. if $v_j^B > v_j^A - \frac{p_j}{k_j}$ then y_j optimal is $y_j^{opt} = Y_j$

In figure 3 y_j is fixed and Y_j- variable, being represented those situations when for every level of demand obtained income is equal or lower than zero, and the same case when income is greater than 0. These situations depends by sign between relations $p_j / (p_j + v_j^A)$ and k_j . In figure 3 is represented behavior of function $\varphi_j(y_j, Y_j)$ in dependence of relations between y_j and Y_j .

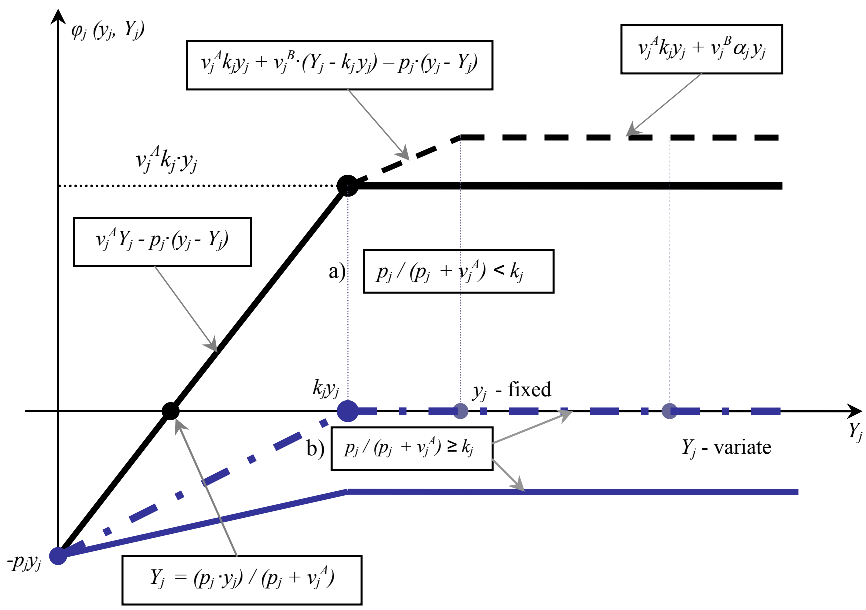

Figure 3. Dependence φ_j (y_j, Y_j) of demand Y_j (y_j – is fixed, y_j >

If $Y_j = k_j y_j$ then we obtain the following form of objective function:

$$\varphi_j = v_j^A k_j y_j - p_j(y_j - Y_j) = v_j^A k_j y_j - p_j y_j + p_j k_j y_j = y_j(v_j^A k_j + p_j(k-1)) = y_j \cdot (v_j^A k_j - p_j \alpha_j)$$

that represents maximal income from selling of good *j* of quality A, when supply of this good is y_j units.

$$v_j^A k_j - p_j \alpha_j = 0 \Leftrightarrow v_j^A k_j = p_j(1 - k_j) \Leftrightarrow v_j^A k_j - p_j k_j = p_j \Leftrightarrow k_j = \frac{p_j}{v_j^A + p_j}$$

There are two cases, if:

a) $\frac{p_j}{v_j^A + p_j} < k_j$ then $\max_{Y_j} \varphi_j = (v_j^A k_j - p_j \alpha_j) \cdot y_j > 0$ while $Y_j \ge k_j y_j$;

b) $\frac{p_j}{v_j^A + p_j} \geq k_j$ then $\max_{Y_j} \varphi_j \leq 0$ so, is not reasonable supply product *j*.

Further, will describe a method of solve of obtained model at the third generalization. Method of projection of generalized gradient is based on concept of subgradient and offers an approximate solution (with admissible error).

Let us admit, first of all that demand Y has a deterministic nature, Y=const.

Evident, relations (8) determine a set of vectors *y*, that belongs to a N-dimensional parallelepiped and will be noted with ***D***. So, constraints (8) can be represented as $y \in D$.

To solve the model (5) - (8) using method of projection of generalized gradient, will introduce a function, that replace constraints (7):

$$\Phi(y) = \max\left\{\overline{\Phi}_1(y), \overline{\Phi}_2(y), ..., \overline{\Phi}_i(y), ...\overline{\Phi}_m(y)\right\} \leq 0 \qquad (9)$$

and $\overline{\Phi}_j(y) = \sum_{j=1}^{n} a_{ij} y_j - b_j \leq 0 \quad i = \overline{1, m}$

Relation (9) is true if and only if relation (7) is true. Function $\Phi(y)$ determines maximal value of deviation of restriction. If this value is positive, result that at least a constraint is not satisfied and function will indicate maximal deviation.

Idea of above mentioned method consist in following. There is generated a set of points $y^0, y^1, ..., y^k, y^{k+1}...$, initial point y^0 it is given, and is chosen by user (decision maker) in dependence of problem particularities. Having approximation y^k, a new (next) approximation y^{k+1} is determined as:

$$y^{k+1} = \Pi_D(\overline{y}^k) \qquad (10)$$

that is, we do operation of project of vector $\overline{y}$ on set *D*, that is compute by relation:

$$\overline{y}^k = y^k + h_k \cdot \eta^k,$$

where η^k - represents direction of motion and h_k - step length. In order to converge to solution, series $h_0, h_1, ..., h_k, ...$ must satisfy following constraints:

$$h_k > 0, h_k \to 0, \sum_{k=0}^{\infty} h_k = 0 \qquad (11)$$

For this method, series h_k is calculated by formula $h_k = \frac{H}{k+1}$, $H > 0$ and will satisfy restrictions (11).

Vector of motion direction η^k is constructed by formula:

$$\eta^k = \begin{cases} subgrad(V(y^k, Y)), & \Phi(y^k) \leq 0 \\ -subgrad(\Phi(y^k)), & \Phi(y^k) > 0 \end{cases} \qquad (12)$$

In situation when vector $\eta^k = (\eta_1, \eta_2, ..., \eta_j, ..., \eta_n)$ is calculated by relation $\eta^k = subgrad(V(y^k, Y))$, then its elements can be compute by formula $\eta_j = \eta_j^1 + \eta_j^2 + \eta_j^3$, where:

$$\eta_j^1 = \begin{cases} v_j^A \cdot k_j & if \quad k_j y_j^k < Y_j \\ 0 & if \quad k_j y_j^k \geq Y_j \end{cases}$$

$$\eta_j^2 = \begin{cases} 0 & if \quad Y_j \leq k_j y_j^k \\ -v_j^B k_j & if \quad k_j y_j^k < Y_j \leq y_j^k \\ v_j^B \alpha_j & if \quad y_j^k < Y_j \end{cases}$$

$$\eta_j^3 = \begin{cases} -p_j & if \quad y_j^k - Y_j \geq 0 \\ 0 & if \quad y_j^k - Y_j < 0 \end{cases}$$

Elements of vector determinate by $subgrad(\Phi(y^k))$ represents coefficients a_{ij}, where $j = \overline{1,n}$, and *i* depends of number of constraint that maximize function $\Phi(y)$.

New obtained approximation by formula (5), we get through projection of vector $\overline{y}$ on set ***D***. This operation of projection takes place as follows:

$$y_j^{k+1} = \begin{cases} y_j^{\min}, \ if \quad \overline{y}_J^K < y_j^{\min} \\ y_j^{\max}, \ if \quad \overline{y}_J^K > y_j^{\max} \\ \overline{y}_J^K, \quad contrarily \end{cases} \qquad j = \overline{1,n}$$

Process of construction of series continues while *k* is lower than a fixed number by user, just this moment guarantees a finite number of steps. By this value depends number of iterations and it must be significant (>10^4).

Of course, there is no guarantee that solution (approximation) obtained at the last iteration is the best; a better one can be at previous or next iteration but differs unessential.

To use "method of projection of generalized gradient" besides of information from model (number of constraints – *m*, number of variables – *n*, vectors v^A, v^B, Y, *k*, *p*, y^{max}, y^{min}, *b*, matrix *A*) we must get following parameters – number of iteration, interval of showing of solutions and the most important initial vector y^0.

Further, will succinctly describe aspects of some numerical methods [3] based the same on notion of subgradient, to solve decisional models like (1)-(3) for situations when factor of demand Y is stochastic or uncertain. In case of stochastic model [5] will consider objective function:

$$R_{stochastic}(y) = E_Y\left[V(y, Y)\right] \qquad (13)$$

Objective function of model where demand is uncertain will be defined in Wald aspect (minmax):

$$R_{uncertain}(y) = \min\left[V(y, Y)\right] \qquad (14)$$

In both situations as directions of movement are used random vectors that depend of simulations of demand.

In case of stochastic model vector of direction of movement on *k* iteration:

$$\eta^k = \begin{cases} subgrad(V(y^k, Y^k)), & if \quad \Phi(y^k) \leq 0 \\ -subgrad(\Phi(y^k)), & if \quad \Phi(y^k) > 0 \end{cases}$$

Here Y^k represents an independent observation (a new simulation) of aleatoric factor Y in correspondence with given low of demand distribution.
In situation when Y is uncertain, set of possible states of Y is defined as a probabilistic measure P(dY), and element Y^k at iteration k is defined as follow:

$$Y^k = \begin{cases} Y^{k-1}, \; if \;\; V(y^k, Y^{k-1}) \geq V(y^k, \widetilde{Y}^k) \\ \widetilde{Y}^k, \; for \;\; V(y^k, Y^{k-1}) < V(y^k, \widetilde{Y}^k) \end{cases}$$

Set $\widetilde{Y}^0, \widetilde{Y}^1, ..., \widetilde{Y}^k, ...,$ represent independent observation of factor of demand *Y* in correspondence with given low of demand distribution *P(dY)*.

Some experimental results (for the first generalization)

There are considered the follow problem. A firm can product two kind of goods. Technological matrix $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.5 & 0.3 \end{bmatrix}$; vector of available resources $b = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$; vector of revenue per unit of product *(v_1, v_2)* = (0.5; 1.5); the cost per unit of product *(p_1, p_2)* = (0.25; 0.8); The minimal quantity of supply y_{min}=(0;0), and the maximal y_{max}=(20;20).

Firstly, simplex method can not be adapted for such model, indifferent of decisional situations (certain, risk or uncertain). Secondly, the model is not linear and non-differentiable, this situation been caused by relation between possibilities of output of product system and behavior of demand.

Remark. The analyzed model can be reduced at classical linear model (for certain, risk or uncertain) if quantities of demand for all sorts of goods, excel significant the capacities of system.

Further, will be analyzed the iterative behavior of supply and corresponding revenue in dependence of nature of demand, that can be deterministic, risky or uncertainly.

Deterministic case (demand is known).

Let there be given the demand Y=(3; 7) that belong to admissible domain. We considered two situations:

1. Point of start y^0= (0; 0)

Numerical results are presented in follow table:

Table 1. Numerical data for deterministic case

k	y_1	y_2	F_i	V_{max}
0	0	0	0	0
5	0	12.3	0	6.259
10	2.58	8.08	0	10.925
15	2.68	6.45	0	11.025
100	3.01	7.02	0	11.910
200	3.00	7.01	0	11.988

At given iteration *k*, F_i represents value $\max\{\Phi(yk);0\}$, where function $\Phi(y)$ is defined in relation (9)

Evident, for such level of supply the firm can not get positive revenue because the output is zero. But simultaneously with increase of number of iterations the quantity of supply approaches to the optimal variant (obvious, equal with demand), and respective income tends to the maximal value.

2. Initial supply y^0= (15; 18)

In situation when, supply essentially exceeds demand, evident, revenue can get negative value. Starting the optimization algorithm of process of production, the vectorial series of consecutive supplies tends to the optimal variant of decision, which is an element of polygon of constraints (in this case studied model is reduced at linear one). This algorithm get with such approximation the same solution obtained using the simplex method.

In these circumstances, producer has to make the maximal possible quantities of goods, been assured that produced volumes will be sold.

Stochastic case

The demand is considered an aleatory vector in such a way that the first its component is a discrete random variable (with given probabilities), and the second component represents a continuous random variable, constant distributed on given interval. Using this information, we simulate different possible values of demand that alternates with iterative searching process of optimal variant of output (considering as objective maximization of average income). Obtained results are selective shown in table 2.

Table 2. Numerical data for situation of risk

k	y_1	y_2	F_i	V_{medium}
0	0	0	0	0
5	0	12.3	0	0
10	2.31	8.08	0	8.728
15	1.44	8.67	0	8.579
100	3.08	6.84	0	8.535
200	3.09	7.77	0	8.812

The following diagram (figure 4) represents dynamic of modification of average income in relation with increase number of iterations *k*. On horizontal axis with points, are marked iterations at which are obtained "fully" inadmissible solutions (when $F_i > \varepsilon$; $\varepsilon=10^{-2}$). At iterations with admissible decisions ($F_i \le \varepsilon$) we see an increase tendency of average revenue.

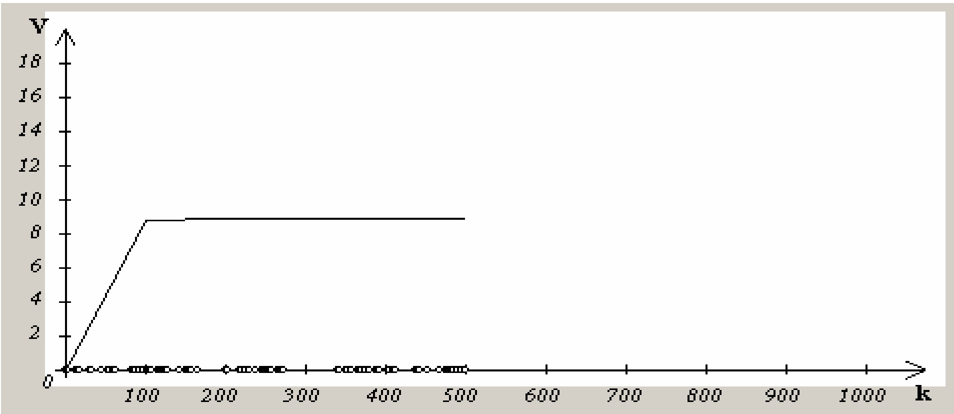

Figure 4. Graphical representation of dynamic of average income

Uncertain situation

The case is similar with cu stochastic variant (see figure 5), demand is manifested with the same possible values. The optimal decision of output level is taken in accordance with Wald criteria (maxmin).

In these circumstances corresponding algorithm is represented by following data.

Table 3. Numerical data for uncertain case

k	y_1	y_2	F_i	V_{max}
0	0	0	0	0
5	0	12.3	0	-5.194
10	1	1.96	0	2.704
15	0.38	2.99	0	2.158
100	0.001	2.06	0	2.991
200	0.003	2.11	0	2.951

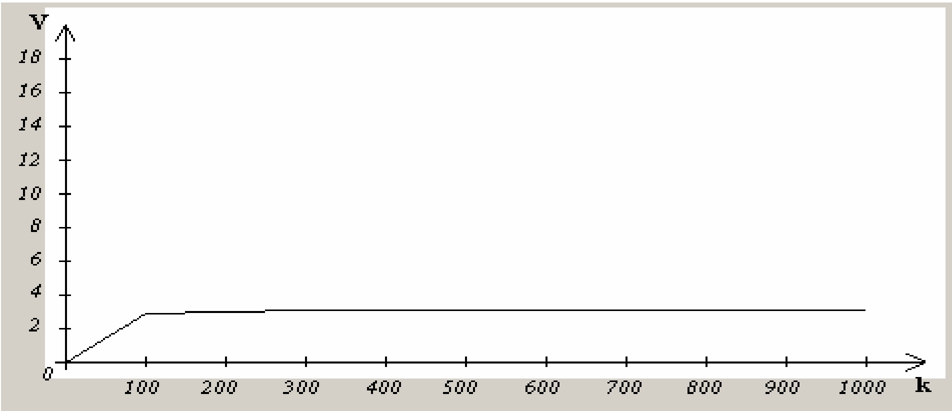

Figure 5. Graphical representation of revenue modification (Wald criteria)

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