METRICS ON ENTITIES SPACES

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Abstract: *In this paper, we deal with concept of "metric". At first, we briefly discuss some issues regarding this very shaded notion in human knowledge. Secondly, we emphasize its usefulness in Mathematics, particularly in the relatively recent field of fuzzy models.*

Key words: *algebraic structure; vectorial space; metric; norm; fuzzy space; fuzzy number; fuzzy random variable*

1. Introduction

"Metric" is one of the words with great spreading and equally many senses, depending on the scope we think about. For instance, in computer networking, it characterizes a way (or a route), while in general relativity theory it describes the spacetime complex, in software it's one important tool for pointing out certain characteristics. In connection with this last assertion, an interesting investigation regarding text entities and adjacent evaluation algorithms (among the text characteristics, "orthogonality" has received great attention) are presented in [9][1]. However, in this paper, we focus our attention on some categories of metrics and their utility in the field of mathematics. In fact, it's the domain in which this concept proves its true value. It is sufficient to enumerate geometry, algebraic theory and fuzzy systems.

2. Metric spaces

Definition 2.1. [11]

Consider a set $X \neq \phi$.

A function $d : X \times X \to R$ is called metric or distance if:

i) $d(x, y) = d(y, x), \forall x, y \in X$;

ii) $d(x, y) \geq 0, \forall x, y \in X$ and

$d(x, y) = 0 \Leftrightarrow x = y$;

iii) $d(x, y) \leq d(x, z) + d(z, y), \forall x, y, z \in X$.

Thus the ordered pair (X, d) is a metric space.

Definition 2.2. [7]

A group is an algebraic structure $(G, \circ), G \neq \phi$, which fulfill the conditions:

i) closure: $a \circ b \in G, \forall a, b \in G$;

ii) "$\circ$" is associative: $a \circ (b \circ c) = (a \circ b) \circ c, \forall a, b, c \in G$;

iii) identity element: $\exists e \in G$ such that $a \circ e = e \circ a = a, \forall a \in G$;

iv) inverse element: $\forall a \in G, \exists a' \in G$ such that $a \circ a' = a' \circ a = e$.

If the commutative rule ($a \circ b = b \circ a, \forall a, b \in A$) is also satisfied, then $(G, \circ)$ is called abelian (or commutative) group.

Definition 2.3. [7]

We define a field as a set $K \neq \phi$ with two binary algebraic (symbolically writed as addition and multiplication) operations

$$+ : K \times K \to K; (a, b) \mapsto a + b$$

$$\cdot : K \times K \to K; (a, b) \mapsto a \cdot b$$

which satisfies the following requirements:

i) $(K, +)$ is abelian group with identity element called zero: $e_{(+)} = 0$;

ii) $(K, \cdot)$ is group with identity element called unity: $e_{(\cdot)} = 1$;

iii) $1 \neq 0$.

If $(K, \cdot)$ is commutative group, then $(K, +, \cdot)$ is a commutative field.

Definition 2.4. [1]

Let $X \neq \phi$ and $(K, +, \cdot)$ commutative field.

We define the following arithmetical operations:

1) $\oplus : X \times X \to X; (x, y) \mapsto x \oplus y$ (the sum of two vectors);

2) $\otimes : K \times X \to X; (a, x) \mapsto a \otimes x$ (scalar multiplication).

We say that (X, K) is a vectorial space if and only if:

i) $(K, \oplus)$ is a commutative group.

ii) $(a + b) \otimes x = (a \otimes x) \oplus (b \otimes x), \forall a, b \in K, \forall x \in X$;

iii) $a \otimes (x \oplus y) = (a \otimes x) \oplus (a \otimes y), \forall a \in K, \forall x, y \in X$;

iv) $a \otimes (b \otimes x) = (ab) \otimes x, \forall a, b \in K, \forall x \in X$;

v) $1 \otimes x = x, \forall x \in X$, where $k \in K$ and $1k = k1 = k, \forall k \in K$.

Example 2.1. [1]

A classical example of vectorial space is (R^n, R) where:

i) $R^n = \{x = (x_1, ..., x_n) \mid x_j \text{ are real numbers for all } j = \overline{1, n}\}$

ii) If $a \in R, x, y \in R^n$ then

$$x + y = (x_1 + y_1, ..., x_n + y_n)$$

and

$$ax = (ax_1, ..., ax_n).$$

Definition 2.5. [11]

Consider (X, K) a vectorial space with dimension equal to n and $(K = R) \vee (K = C)$.

A function $\|\cdot\| : X \to K$ is called norm if the following relations hold:

i) $\|x\| \geq 0, \forall x \in X$ and $\|x\| = 0 \Leftrightarrow x = 0_X$;

ii) $\|ax\| = |a| \cdot \|x\|, \forall a \in K, \forall x \in X$;

iii) $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in X$.

Example 2.2. [11]

For instance, if $x, y \in R^n$ it results that

$$\|x\| = \sqrt{x_1^2 + ... + x_n^2}$$

and

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2} .$$

3. Metrics on fuzzy spaces

Definition 3.1. [8]

A fuzzy subset of a given set X is described by a function $u(x)$ (membership degree of x in X) with $u : X \to [0,1]$. The set $F(x) = \{u \mid u : X \to [0,1]\}$ contain all fuzzy subset previously defined.

Definition 3.2. [8]

The $\alpha - \text{level set}$ of $u(x)$ is defined as

$$L_\alpha(u) = \{x \in X \mid u(x) \geq \alpha\}.$$

Definition 3.3. [8]

If X, Y are two subsets of R^n, the Hausdorff metric between X and Y is given by the formula:

$$d(X,Y) = \max\left\{\sup_{x\in X}\inf_{y\in Y}\|x-y\|, \sup_{y\in Y}\inf_{x\in X}\|x-y\|\right\} =$$

$$= \max\left\{\sup_{x\in X} d(x,Y), \sup_{y\in Y} d(y,X)\right\}.$$

Remark that Hausdorff metric satisfies the requirements given in Definition 2.1, namely symmetry, nonnegativity and triangle inequality. Moreover it is important to be mentioned that the space of subsets of R^n with this distance is a complete space.

It is necessary to remark that this metric plays an important role in some calculation on fuzzy random variable (at short, FRV).

Definition 3.4. [8]

It's possible to define a metric on $F(R^n)$ as follows:

$$d_\infty(u_1,u_2) = \sup_{\alpha\in[0,1]} d(L_\alpha(u_1), L_\alpha(u_2)).$$

One can prove that $(F(R^n), d_\infty)$ is a complete space.

Another way to define a such type of metric is:

$$d_p(u_1,u_2) = \left(\int_0^1 d^p(L_\alpha(u_1), L_\alpha(u_2))d\alpha\right)^{\frac{1}{p}}.$$

Remark 3.1.

An interesting development of a distance between two fuzzy numbers is presented in [5]. First, a fuzzy number x is described as a pair $(\underline{x}(r), \overline{x}(r))$ of two special functions defined on closed interval $[0,1]$. Next, the distance between x and y is measured through a metric (which admit the quality of complectness on the respective fuzzy space) given by:

$$d(x,y) = \left(\int_0^1 (\underline{x}(r) - \underline{y}(r))^2 dr + \int_0^1 (\overline{x}(r) - \overline{y}(r))^2 dr\right)^{\frac{1}{2}}.$$

This metric was useful in the process of managing a regression model with fuzzy data and real parameters.

Remark 3.2.

A special type of fuzzy numbers is LR (left-right); in such a case, we have [5]:

$$u(t) = \begin{cases} L\left(\dfrac{m-t}{a}\right), & t \le m, a > 0 \\ R\left(\dfrac{t-m}{b}\right), & t \ge m, b > 0 \end{cases}$$

where L, R are nonincreasing and nonnegative functions with $L(0) = R(0) = 1$. The triangular form is a particular case of LR form, with many applications in fuzzy statistics.

In [6] two LR *f*-numbers are compared by Hukuhara difference as in the following lines.

Consider $x, y \in F_{LR}$, $x = (u_x, l_x, r_x)_{LR}$, $y = (u_y, l_y, r_y)_{LR}$, (where l and r mean the left spread and the right spread, respectively). Under assumption that $l_x \geq l_y$ and $r_x \geq r_y$, it is possible to define the Hukuhara difference between these two numbers as it follows:

$$x \Theta_H y = (u_x - u_y, l_x - l_y, r_x - r_y)_{LR}.$$

The Hukuhara difference appears in some theoretical results regarding covariance between two fuzzy random variables [6].

Remark 3.3.

As regression models, fuzzy clustering is one field in which a proper choice of a metric is very important, too. A classical model is fuzzy *c*-means [3]. Generally, it is based on searching the minimum of the following function:

$$J(U,V) = \sum_{i=1}^{c} \sum_{k=1}^{N} u_{ik}^{m} \left\| x_k - v_i \right\|^2,$$

where v_i represents the "center" for the *i*'th cluster, x_k the *k*'th data, u_{ik} the membership degree of x_k in the *i*'th group of data, and m is a fuzzification exponent. In dedicated literature, many improvements of this method were published and among them are eFFCM and geFFCM, suitable for managing a large number of data [3].

4. Conclusions

At the end, beyond all doubt we can say that the "metric" is one essential tool in all kinds of (roughly speaking) measurements. It is useful for point out "distances" between somehow abstract things such as mathematical objects and is a vital question in the process of building theoretical pattern which reproduce with more rigour the complex phenomena of nature.

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