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# SOME NOTES ON THE LOGISTIC DISTRIBUTION

**Alexandru ISAIC-MANIU**
PhD, University Professor, Department of Statistics Econometrics
University of Economic Studies, Bucharest, Romania
(**Co)Author of the books:** Proiectarea statistica a experimentelor (2006), Enciclopedia calitatii (2005), Dictionar de statistica generala (2003), Statistica pentru managementul afacerilor (2000), Metoda Weibull (1983)

**E-mail:** al.isaic-maniu@csie.ase.ro, **Web page:** http://www.amaniu.ase.ro

**Viorel Gh. VODĂ**
PhD, Senior Scientific Researcher,
"Gh. MIHOC - C. IACOB" Institute of Mathematical Statistics and Applied Mathematics of the Romanian Academy, Bucharest, Romania
**Co-author of the books:** Proiectarea statistica a experimentelor (2006), Dictionar de statistica generala (2003), Manualul calitatii (1997)

**E-mail:** von_voda@yahoo.com

**Abstract:** *In this paper we study some properties of the density function* $f(\mathrm{x};\theta)=\theta\cdot e^{-\mathrm{x}}\left(1+e^{-\mathrm{x}}\right)^{-(\theta+1)}$, $\mathrm{x}\in R$, $\theta>0$ *which may be obtained from the distribution of the last order statistic from a reduced logistic population* $\mathrm{F}(x)=1/\left(1+\mathrm{e}^{-x}\right)$, $x\in\mathrm{R}$. *Its truncated variant is also discussed. The point and interval estimations for* $\theta$ *are provided also. The so-called* $(\mathrm{P},\gamma)$ *- type statistical tolerances are constructed and a comment on the hazard rate is done also. The last paragraph is devoted to testing procedures on the parameter involved.*

**Key words:** *logistic distribution; Burr-Hatke family; last order statistic; truncation; MLE;* (P, γ) *- type tolerances*

## 1. Introduction

As it is well-known, the Belgian scientist Pierre François VERHULST (1804 - 1849) proposed in 1838 a "demographic growth cure" which was called later as the logistic function:

$$\mathrm{Y}=\frac{\mathrm{A}}{\mathrm{B}+10^{\mathrm{a}+\mathrm{bX}}} \quad \text{or} \quad \mathrm{Y}=\frac{\mathrm{A}}{\mathrm{B}+\mathrm{e}^{\mathrm{a}+\mathrm{bX}}} \tag{1}$$

where $\mathrm{x}\geq 0,\quad \mathrm{a},\mathrm{b}>0,\quad \mathrm{a},\mathrm{b}\in\mathrm{R},$ e being the Euler's number $(\mathrm{e}\approx 2{,}71828)$. The usual form used now in econometric studies is the following :

$$Y = \frac{A}{B + e^{-CX}}, \quad a, B, C > 0, \quad x \geq 0 \tag{2}$$

(for historical details, see Iosifescu *et al*, 1985 [8, pages 280 - 281][1]).

Verhulst's function is obtained from a differential equation of the type $y' = u(y)$, where $y = y(x)$, $y_0 = u(x_0)$, $x_0 \in R$, by taking $u(y) = y - y^2$.

Therefore, we have:

$$\frac{dy}{dx} = y(1-y) \quad \text{or} \quad \frac{dy}{y(1-y)} = \frac{dx}{1} \tag{3}$$

which provides the reduced form of the logistic function:

$$y = \frac{1}{1 + e^{-x}}, \quad x \in R \tag{4}$$

If we consider y(x) as a cumulative distribution function (cdf) of a given random variable $X : y = F(x) = \Pr ob\{X \leq x\}$, the differential equation:

$$\frac{dF}{dx} = F(1-F)g(x), \quad F_0 = F(x_0), \quad x_0 \in R \tag{5}$$

where g(x) is an arbitrary positive function $(x \in R)$, provides - for several choices of g(x) - various cdf(s).

The form (4) generates the well-known BURR-HATKE family of distributions - see Burr (1942 [2]) and Hatke, (1949 [6]). Johnson, Kotz and Balkrishan (1994 [14, page 54]) enlists twelve such cdf(s), denoted from I to XII, the second one being just the Verhulst distribution function (4) if one takes $g(x) \equiv 1$. Notice that the most used cdf from these twelve cdf(s) is XII-th one: $1 - \left(1 + x^c\right)^{-k}, \quad x \geq 0, \quad c, k > 0$ (see Rodriguez, 1977 [16] or Vodă, 1982 [18]).

In this paper, we shall investigate mainly, the density function of the last order statistics from the logistic form (4).

## 2. Preliminaries

If X is a continuous random variable with $F_X(x)$ as a cdf and if $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(k)} \leq X_{(n+1)} \leq \ldots \leq X_{(n)}$ is an ordered sample on X, then, the distribution of the last order statistics $X_{(n)}$ is given as:

$$F_{X_{(n)}}(x) = \Pr ob\{X_{(n)} \leq x\} = \Pr ob\{all\ X_k \leq x\} = F_X^n(x) \tag{6}$$

(see David, 1970, [3 page 38]).

In our case, if $F_X(x) = F(x) = 1/(1 + e^{-x})$, then

$F_{X_{(n)}}(x) = 1/(1 + e^{-x})^n \quad x \in R, \quad n \in N \setminus \{0\}$. It is more interesting to change n by $\theta$ - a positive real parameter and then we obtain the cdf:

$$X : F(x; \theta) = \frac{1}{(1 + e^{-x})^{\theta}}, \quad x \in R, \quad \theta > 0 \tag{7}$$

In a reliability frame we prefer to have $x \in [0,+\infty)$ and hence it is worth to truncate the density of X namely:

$$X : f(x;\theta) = \theta e^{-x}\left(1+e^{-x}\right)^{-(\theta+1)}, \quad x \in R, \quad \theta > 0 \tag{8}$$

as follows:

$$f_T(x;x_T,\theta) = \frac{1}{1-F(x_T;\theta)} \cdot f(x;\theta), \quad x \geq x_T \geq 0, \quad \theta > 0 \tag{9}$$

or

$$f_T(x;x_T,\theta) = \frac{\left(1+e^{-x_T}\right)^{\theta}}{\left(1+e^{-x_T}\right)^{\theta}-1} \cdot \theta \cdot e^{-x}\left(1+e^{-x}\right)^{-(\theta+1)}, \quad x \geq x_T \geq 0, \quad \theta > 0 \tag{10}$$

If the truncation point is $x_T = 0$, then:

$$f_T(x;x_T,\theta) = \frac{1}{1-1/2^{\theta}} \cdot \theta \cdot e^{-x}\left(1+e^{-x}\right)^{-(\theta+1)}, \quad x \geq 0, \quad \theta > 0 \tag{11}$$

## 3. Estimation problems

First, let us notice that even for the reduced form (4) the method of moments is difficult to be applied. If we consider the form (11) consider the integral:

$$m_k(x) = \int_0^x t^k f_T(t)dt \tag{12}$$

which provides the equation (the derivative of (12)):

$$m'_k(x) = x^k f_T(x) \tag{13}$$

and the k - th moment is given as:

$$E(X^k) = \int_0^\infty x^k f_T(x)dx \tag{14}$$

It is easy to see now that $E(X) = m_1(\infty) - m_1(0)$ with $m_1(0) = 0$ and the problems lies in the approximation of $m_1(\infty)$.

The MLE - maximum likelihood estimation - method gives straightforward results in the case of (7). We have the likelihood function

$$L(x_1,x_2,\ldots x_n;\theta) = \theta^n \cdot \left[\exp\left(-\sum_1^n x_i\right)\right]\prod_1^n\left(1+e^{-x_i}\right)^{-(\theta+1)} \tag{15}$$

where $x_1,x_2,\ldots x_n$ is a random sample on X. After some simple algebra, we obtain the MLE for $1/\theta$ as:

$$\left(\widehat{\frac{1}{\theta}}\right) = \frac{1}{n}\sum_1^n \ln\left(1+e^{-x_i}\right) \tag{16}$$

The case of the truncated variable (11) provides successively:

$$L(x_1,x_2,\ldots x_n;\theta) = \frac{2^{n\theta}}{\left(2^{\theta}-1\right)^n} \cdot \theta^n \cdot \left[\exp\left(-\sum_1^n x_i\right)\right] \cdot \prod_1^n\left(1+e^{-x_i}\right)^{-(\theta+1)} \tag{17}$$

$$\ln L = n\theta \ln 2 - n\ln\left(2^{\theta}-1\right) + n\ln\theta - \sum_{1}^{n} x_i - (\theta+1)\sum_{1}^{n}\ln\left(1+e^{-x_i}\right) \tag{18}$$

$$\frac{\partial \ln L}{\partial \theta} = n\ln 2 - \frac{n\cdot 2\ln 2}{2^{\theta}-1} + \frac{n}{\theta} - \sum_{1}^{n}\ln\left(1+e^{-x_i}\right) \tag{19}$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta} - \frac{n\ln 2}{2^{\theta}-1} - \sum_{1}^{n}\ln\left(1+e^{-x_i}\right) = 0 \tag{20}$$

The likelihood equation (20) is of the following type:

$$\varphi(u) = \frac{A}{u} - \frac{B}{2^{u}-1} - C = 0 \tag{21}$$

where A, B, C > 0 and it is clear that it has a solution science $\lim\limits_{u\to 0^{+}}\varphi(u) = +\infty$ and $\lim\limits_{u\to+\infty}\varphi(u) = -C$.

Numerical methods are needed to approximate (u).

We shall state now the following.

**Proposition**. If X is a logistic random variable with cdf given by (7), then the variable $Y = \ln\left(1+e^{-X}\right)$ is exponentially distributed and the consequences are:

(i) the MLE for $1/\theta$ is unbiased and with minimum variance;

(ii) the distribution of $\left(\frac{\hat{1}}{\theta}\right)$ is a Gamma one;

(iii) the statistic $U = 2n\theta/\hat{\theta}$ has a Chi - square $\left(\chi^2\right)$ distribution with 2n degrees of freedom.

**Proof**.We have immediately:

$$\begin{aligned} F(z) &= \Pr ob\left\{\ln\left(1+e^{-x}\right) \le z\right\} = 1 - \Pr ob\left\{x < \ln\left(e^{z}-1\right)\right\} = \\ &= 1 - \theta \int_{-\infty}^{\ln\left(e^{z}-1\right)} e^{-x}\left(1+e^{-x}\right) dx \end{aligned} \tag{22}$$

Taking into account that

$$\left(\int_{a(z)}^{b(z)} \varphi(x)dx\right)' = b'(z)\cdot\varphi[b(z)] - a'(z)\cdot\varphi[a(z)] \tag{23}$$

we obtain from (22):

$$F'(z) = \theta \exp(-\theta z) \text{ with } z \ge 0, \quad \theta > 0 \tag{24}$$

Since

$$E\left[\left(\frac{\hat{1}}{\theta}\right)\right] = \frac{1}{\theta} \quad \text{and} \quad Var\left[\left(\frac{\hat{1}}{\theta}\right)\right] = \frac{1}{n\theta^2} \tag{25}$$

the property (i) is proved.

Now, since each variable is exponentially distributed, then its sum is Gamma distributed (see Johnson et al, 1994 [14, page 337 and page 494]). The density of $\left(\frac{\hat{1}}{\theta}\right)$ is therefore

$$\varphi\left[\left(\frac{\hat{1}}{\theta}\right);\theta,n\right] = \frac{n^n\theta^n}{\Gamma(n)}\left(\frac{\hat{1}}{\theta}\right)^{n-1} \cdot \exp\left\{-n\theta\left(\frac{\hat{1}}{\theta}\right)\right\} \tag{26}$$

where $\left(\frac{\hat{1}}{\theta}\right) > 0, \quad \theta > 0, \quad n \in N \setminus \{0\} \text{ and } \Gamma(n)$ is the Gamma function; here we have $\Gamma(n+1) = n!$ (the property (ii) is also proved).

To demonstrate (iii), we may write the characteristic function of U, and we have:

$$\varphi_U(t) = E\left(e^{itu}\right) = E\left[\exp\left(i(2\theta t)\sum_{j=1}^{n}\ln\left(1+e^{-xj}\right)\right)\right] = \prod_{j=1}^{n}\left(1-\frac{1}{\theta}2\lambda t\right)^{-1} = (1-2it)^{-n} \tag{27}$$

which is just the characteristic function of a Chi-square variable with 2n degrees of freedom (see for instance Wilks, 1962 [20 page 86]).

Based on this property, one can construct confidence intervals of minimum length (L) for $\theta$. Namely, we have to determine two limits $L_{inf}$ and $L_{sup}$ such that

$$\Pr ob\left\{L_{inf} < 2n\theta/\hat{\theta} < L_{sup}\right\} = 1-\alpha \tag{28}$$

where $0 < \alpha < 1, \quad \alpha$ - given, with the property (29) $L = \frac{1}{2Y_0}\left(L_{sup} - L_{inf}\right) =$ minimum, where $Y_0 = \sum_{1}^{n}\ln\left(1+e^{-x_i}\right)$.

In accordance with Tate and Klett`s results (1959, [17]) the following system has to be solved:

$$\int_{L_{inf}}^{L_{sup}} u^{n-1}e^{-u/2}du = 2^n(1-\alpha)\Gamma(n); \quad \left(\frac{L_{sup}}{L_{inf}}\right)^{n-1} = \exp\left(\frac{L_{sup}-L_{inf}}{2}\right) \tag{30}$$

The solutions ($L_{inf}$, $L_{sup}$) are founded by entering Tate - Klett`s tables in the cell corresponding to 2n degrees of freedom (see also Isaic-Maniu and Vodă, 1989 [10]).

## 4. (P, $\gamma$) - type tolerance limits

The literature devoted to the problem of statistical (or "natural") tolerance is very wide. In 1981, Miloš Jílek (Prague) compiled a large bibliography on this subject matter (see Jílek [12]) and in 1988 the same author (see Jílek [13]) provided a sound monograph with applications (chapter 12, pages 198 - 235 of [13]) of these tolerances.

Origin of this concept - which goes back to W. A. Shewhart (1891 - 1967), Samuel S. Wilks (1906 - 1964), Abraham Wald (1902 - 1950), Herbert Robbins (1915 - 2001) - and some other (historical aspects may be found in [7]).

The mathematical formulation of the problem is the following: if X is a continuous random variable defined on $D \subseteq R$ and having a density function $f(x;\theta)$, then we have to construct two statistics $T_L$ and $T_U$ (lower and upper) such that at least a proportion (P) of this population {X} will be found between $T_L$ and $T_U$, and this must happen with a given probability $\gamma(0<\gamma<1)$, that is.

$$\operatorname{Prob}\left\{\int_{T_L}^{T_u} f(x;\theta)dx \geq P\right\} = \gamma \tag{31}$$

where $0 < P, \gamma < 1$ are previously chosen. These elements $T_L = \varphi(x_1, x_2, \ldots x_n)$ and $T_U = \Psi\ x_1, x_2, \ldots x_n\ )$where $\{x_i\}_{1\le i\le n}$ are sample values on X are called $(P,\gamma)$ - type tolerance limits. In a reliability context one is interested in a lower tolerance limits since $X : D \equiv [0,+\infty)$ and we need that at most a proportion (1 - P) of the population to lie between 0 and $T_L$.

In our logistic case, we shall write:

$$\operatorname{Prob}\left\{\int_{T_L}^{+\infty} f(x;\theta)dx \geq P\right\} = \gamma \tag{32}$$

Two ways to deduce $T_L$ will be presented.

**a)** ***The case of large sample*s** In this situation, we may state that the statistic

$$Y = \frac{\left(\frac{\hat{1}}{\theta}\right) - \frac{1}{\theta}}{\frac{1}{\theta\sqrt{n}}} \tag{33}$$

where $\left(\frac{\hat{1}}{\theta}\right)$ is the MLE of $1/\theta$, is approximately normally distributed with E [Y] = 0 and Var [Y] = 1.

The relationship (32) may be written as.

$$\operatorname{Prob}\left\{1 - \left(1 + e^{-T_L}\right)^{-\theta} \geq P\right\} = \gamma \tag{34}$$

or

$$\mathrm{Prob}\left\{\ln(1-P) \geq -\theta \ln\left(1+e^{-T_L}\right)\right\} = \gamma \tag{35}$$

$$\mathrm{Prob}\left\{\frac{1}{\theta} \leq \frac{\ln\left(1+e^{-T_L}\right)}{-\ln(1-P)}\right\} = \gamma \tag{36}$$

which may be rearranged as follows:

$$\mathrm{Prob}\left\{\frac{\left(\frac{\hat{1}}{\theta}\right)-\frac{1}{\theta}}{1/\theta\sqrt{n}} \leq \frac{\frac{\ln\left(1+e^{-T_L}\right)}{-\ln(1-P)}-\frac{1}{\theta}}{1/\theta\sqrt{n}}\right\} = \gamma \tag{37}$$

(we did replace $1/\theta$ in (36) by its MLE).

The right-hand side of the inequality in (37) is just the $\gamma$ - quantile of N (0, 1) distribution - let it be $u_\gamma$ - and hence we may write:

$$\frac{\ln\left(1+e^{-T_L}\right)}{-\ln(1-P)} = \left(\frac{\hat{1}}{\theta}\right)\left(1+\frac{u_\gamma}{\sqrt{n}}\right) \tag{38}$$

or

$$\ln\left(1+e^{-T_L}\right) = \left(\frac{\hat{1}}{\theta}\right)\left(1+\frac{u_\gamma}{\sqrt{n}}\right)\ln\frac{1}{1-P} \tag{39}$$

Since $0 < e^{-T_L} < 1$, then we could use the approximation of ln (1+$x$), 0 < $x$ < 1 given in Abramowitz and Stegun (1964, [1]) to find a polynomial equation in $T_L$ :

$$\ln(1+x) = a_1 x + a_2 x^2 + \ldots + a_5 x^5 + \varepsilon(x) \tag{40}$$

where the error $\varepsilon(x)$ is $|\varepsilon(x)| < 10^{-5}$ and $a_1 \approx 0.99949556$ a.s.o. One may restrict the approximation to the roughest one, namely:

$$\ln\left(1+e^{-T_L}\right) \approx a_1 \cdot e^{-T_L} \tag{41}$$

and hence $T_L$ may be deduced easily by taking logarithms in (39).

We did call this method to find statistical tolerances as "normalizing" one (see Isaic-Maniu and Vodă, 1981, [9], and 1993, [10].

***b) The general case***. When we have an arbitrary sample size, we may re-write (35) as follows:

$$\mathrm{Prob}\left\{\frac{2n\theta}{\hat{\theta}_{MLE}} \leq \frac{1\ln\frac{1}{1-P}}{\ln\left(1+e^{-T_L}\right)} \cdot \sum_{i=1}^{n} \ln\left(1+e^{-x_i}\right)\right\} = \gamma \tag{42}$$

Since $2n\theta/\hat{\theta}_{MLE}$ is Chi-square $\left(\chi^2\right)$ distributed with 2n degrees of freedom, the righ-hand side in the inequality of (42) is just the $\gamma$ - quantile of the $\left(\chi^2\right)$ distribution - let it be $\left(\chi^2{}_{2n;\gamma}\right)$. Therefore, a similar equation with (39) is obtained:

$$\ln\left(1+e^{-T_L}\right)=\frac{1}{\chi^2{}_{2n;\gamma}}\cdot\ln\frac{1}{1-P}\cdot\sum_{i=1}^{n}\ln\left(1+e^{-x_i}\right) \tag{43}$$

## 5. A discussion on the hazard rate

If X is a continuous random variable representing the time-to-failure of a certain device, then the hazard (or failure) rate associated to X is given as:

$$h(x)=\frac{f(x)}{1-F(x)} \tag{44}$$

where F(x) is the cdf of X and $F'(x)=f(x)$ is the corresponding density. The variable X is assumed to be positive.

In our case we have to work with the truncated variable. Suppose that our logistic is truncated at x = 0, that is $x\in[0,+\infty)$. Therefore:

$$F_T(x;\theta)=\frac{1}{1-F(0;\theta)}\int_0^x F(u;\theta)du=K\cdot\left[F(x;\theta)-F(0,\theta)\right] \tag{45}$$

where $K=2^\theta/\left(2^\theta-1\right)$ and $F(0;\theta)=1/2^\theta$.

In this situation, we have:

$$h_T(x;\theta)=\frac{f_T(x;\theta)}{1-F_T(x;\theta)}=\frac{\frac{2^\theta}{2^\theta-1}\cdot\theta e^{-x}\left(1+e^{-x}\right)^{-(\theta+1)}}{1-\left[\frac{1}{\left(1+e^{-x}\right)^\theta}-\frac{1}{2^\theta}\right]} \tag{46}$$

or

$$h_T(x;\theta)=\frac{\frac{\theta\cdot 2^\theta}{2^\theta-1}e^{-x}\cdot\frac{1}{\left(1+e^{-x}\right)^{\theta+1}}}{\frac{2^\theta-1}{2^\theta}-\frac{1}{\left(1+e^{-x}\right)^\theta}} \tag{47}$$

The study of $h_T(x;\theta)$ may be performed denoting $e^{-x}=y$ (we have 0 < y < 1) and consequently, we get.

$$h_T(y;\theta)=\frac{K\theta y\cdot\frac{1}{(1+y)^{\theta+1}}}{\frac{1}{K}-\frac{1}{(1+y)^\theta}} \tag{48}$$

or

$$h_T(y;\theta)=\frac{K^2\theta y}{\left[(1+y)^{\theta}-K\right](1+y)},\quad 0<y<1 \tag{49}$$

An interesting situation arises when we consider the reduced form of the logistic $F(x)=1/(1+e^{-x}),\quad x\in R$ and write formally:

$$h(x)=\frac{\frac{dF}{dx}}{1-F}=\frac{F(1-F)}{1-F}=F(x) \tag{50}$$

In this case, the hazard rate coincides with the distribution function and the behaviour of h(x) is now obvious.

## 6. Testing a simple statistical hypothesis

Since only one parameter is involved in $F(x;\theta)$, we may make use of the fact that

$$U=\frac{2n\theta}{\hat{\theta}_{ML}}=2\theta\cdot\sum_{i=1}^{n}\ln\left(1+e^{-x_i}\right) \tag{51}$$

is Chi-square distributed with 2n degrees of freedom and hence to test:

$$h_0:\theta=\theta_0 \text{ versus } h_1:\theta=\theta_1 \quad (\theta_0<\theta_1) \tag{52}$$

the critical value will be $\chi^2_{2n;\alpha}$ ($\alpha$ - quantile of the $\chi^2$ distribution with 2n degrees of freedom).

More economical is to apply SPRT (Sequential Probability Ratio Test) of Abraham WALD (1902 - 1950) - see his well - known book [ 19 , Chapter 3, pages 37 - 54].

We shall write straightforwardly the logarithm of likelihood ratio ($r_n$):

$$\ln r_n=n\cdot\ln\frac{\theta_1}{\theta_0}+(\theta_0-\theta_1)\cdot\sum_{i=1}^{n}\ln\left(1+e^{-x_i}\right) \tag{53}$$

where $x_i(i=1,2,\ldots,n,\ldots)$ is the sequential sample.

In accordance with Wald`s rules [19, page 49] we could take the following decisions:

a) If

$$\sum_{i=1}^{n}\ln\left(1+e^{-x_i}\right)\geq\frac{\ln A}{\theta_1-\theta_0}+n\cdot\frac{\ln\frac{\theta_1}{\theta_0}}{\theta_1-\theta_0} \tag{54}$$

we accept ($H_0$) and automatically reject the alternative ($H_1$):

b) If

$$\sum_{i=1}^{n}\ln\left(1+e^{-x_i}\right)\leq\frac{\ln B}{\theta_1-\theta_0}+n\cdot\frac{\ln\frac{\theta_1}{\theta_0}}{\theta_1-\theta_0} \tag{55}$$

then reject the null hypothesis ($H_0$) and accept ($H_1$)

c) If

$$\frac{\ln B}{\theta_1-\theta_0}+n\cdot\frac{\ln\frac{\theta_1}{\theta_0}}{\theta_1-\theta_0}<\sum_{i=1}^{n}\ln\left(1+e^{-x_i}\right)<\frac{\ln A}{\theta_1-\theta_0}+n\cdot\frac{\ln\frac{\theta_1}{\theta_0}}{\theta_1-\theta_0} \quad (56)$$

then, the procedure continues by taking the next observation.

Here $A=(1-\beta)/\alpha$ and $B=\beta/(1-\alpha)$, $\alpha,\beta$ being the classical statistical risks in the theory of hypothesis testing (see Wilks, 1962 [20]).

The OC - function (the Operative Characteristic) of the test is given by:

$L(\theta)=\left(A^h-1\right)/\left(A^h-B^h\right)$ where h is the solution of Wald`s equation $E\left(e^{th}\right)=1,\ h\neq 0,\ z=\ln\frac{f(x;\theta_1)}{f(x;\theta_0)}$.

In our case:

$$z=\ln\frac{\theta_1}{\theta_0}+(\theta_0-\theta_1)\cdot\ln\left(1+e^{-x}\right) \quad (57)$$

and consequently:

$$e^{zh}=\left(\frac{\theta_1}{\theta_0}\right)^n\cdot\left(1+e^{-x}\right)^{h(\theta_0-\theta_1)} \quad (58)$$

and

$$E\left[e^{zh}\right]=\left(\frac{\theta_1}{\theta_0}\right)^h\cdot\int_{-\infty}^{+\infty}e^{-x}\left(1+e^{-x}\right)^{h(\theta_0-\theta_1)-\theta-1}dx=1 \quad (59)$$

Imposing the condition $\theta>h(\theta_0-\theta_1)$, we get a parametric representation of the OC - function as

$$\theta=\frac{h(\theta_1-\theta_0)}{\left(\frac{\theta_1}{\theta_0}\right)^h-1} \quad (60)$$

The ASN (Average Sample Number) needed to perform SPRT is given be

$E_\theta(n)=\{L(\theta)\cdot\ln B+[1-L(\theta)]\ln A\}/E_\theta(z)$

where - in one case:

$$E_\theta(z)=\ln\frac{\theta_1}{\theta_0}+(\theta_0-\theta_1)\cdot\frac{1}{\theta}$$

Therefore, the sequential test is completely constructed. We did not re-stated the whole theory - all details are given in Wald [19], Dixon and Massey (1972, [4, pages 300 - 312]) or in more recent works such as those of Govindarajulu (2000 [5]) or Pham (2006, [15]).

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