ABOUT THE IMPOSSIBILITY THEOREM FOR INDICATORS AGGREGATION

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Abstract: *This paper presents some aspects related to the issues of aggregating economic indicators. Departing from the research of Gh. Paun (1983) we will prove a theorem which states that under certain, natural assumptions, it is impossible to obtain an optimum aggregation. Unlike the original work of Paun, in this paper we are giving a rather simple proof of this theorem.*

Key words: *economic indicators; aggregate indicator; optimum aggregation; sensitive indicator; unexplosive indicator; system of indicators*

1. Statistic Indicators. Overview

Due to the complexity of the economic phenomena, the use of a single indicator for describing them is often useless. At the same time, an important component of the economic phenomena evaluation process is represented by the comparison and classification models (that submit to the comparing process of the economic agents, countries, processes, etc).
The speciality literature generally accepts the following features for the indicators[1]:

- The indicators are almost always used in comparisons
- In many situations the indicators are oriented towards action: adequate processes are planed precisely in order to modify the values of the indicators for a specific purpose.

It is widely accepted the fact that the economic domain imposes the use of some multidimensional indicators for characterizing various phenomena.

In order to quantify the performance of a portfolio in the capital market we use the average or expected rate of return as well as its risk, measured through the dispersions and covariance of the components. In order to describe the state of a national economy we use an entire list of macroeconomic indicators: gross domestic product, national product, etc.

The multidimensionality of the indicators used in such situations comes in contradiction with the two ideas referring to indicators, stated above. It is, indeed, difficult to compare two portfolios using the bi-dimensional indicator return-risk, the motivation being represented by simple linear algebra: the multidimensional vectorial spaces cannot be endowed with a relation of total order, i.e. we cannot compare two vectors of this nature.

In the particular situation presented, we can use other comparison methods, for example at the same risk level, we prefer the portfolio with a higher rate of return, and at the same rate of return we prefer the portfolio with a lower risk level.

In parallel, we use the criterion offered by the variation coefficient, as division between risk and expected return; according to this criterion we prefer portfolios with lower values of the variation coefficient.

On the other hand, in practice, other situations might appear such as: processes launched for the quality improvement of an indicator lead to unwanted consequences for the other indicators.

According to the statements above, it is imposed the reduction of the indicator systems dimensions, in such a manner that the reduced system contains at least the same quantity of information as individual indicators, while having the advantage of being more synthetic and easily manageable.

There are mainly two methods to reduce the number of indicators: selection and aggregation.

The selection reduces the number of indicators using statistic methods and techniques. If $S=\{i_1,...,i_n\}$ is the system of primary indicators, then the dimensional reduction can be performed by eliminating redundant information. If between two indicators there is a well defined functional relation, $i_k = f(i_j), k \neq j$, then there is no sense in including in the system both indicators, since the information held by one can be found in the other. For example, if the linear correlation coefficient is positive, $r_{kj} > 0$, it is redundant to use both indicators. This functional dependence can be tracked, for example, using regression techniques.

Another selection type can be accomplished by using elements from the classification theory; the system of indicators is divided into classes, using various criteria of maximizing the distance between classes and minimizing the distance between the elements in the same class (with the purpose of creating classes as homogeneous as possible).[2] Then there are chosen representative indicators from each class, the system formed by these representative indicators being used in the setting of the indicators system.

The primary indicators aggregation implies the building of a new indicators (called aggregated indicator), seen as a function between the initial indicators; if $S=\{i_1,...,i_n\}$ is the system of primary indicators, then the aggregated indicator can be seen as a function $f:\mathbb{R}^n \to \mathbb{R}$, $I_a = f(i_1, i_2, ..., i_n)$.

Generally, in order to reduce the dimensions of the indicators systems we recommend the successive usage of selection and aggregation.

Through aggregation synthetic indicators are being produced, the disadvantage however is that in this case information are being lost. Indeed, through aggregation, the entropy of the indicators system is decreasing, which implicitly leads to a decrease in the quantity of information.

Being given S1 and S2, two subsystems of indicators of the initial system S; we can establish, in certain conditions, the relationship between the system total entropy and the entropy of the component subsystems.[3] Thus, we may have the following cases:

- If S1, S2 are independent, then H(S)=H(S1)+H(S2); in this case we are dealing with a preservation in the quantity of information from the system to its components;
- If S1 conditions S2 (the inverse situation is analogous), then $H(S)=H(S1)+H(S2/S1)\leq H(S1)+H(S2)$; in this case, the quantity of information contained by the total system is smaller than the quantity of information given separately by the individual subsystems.
- If S1, S2 condition one another, then we have a situation analogous to the one discussed above: $H(S)=H(S1/S2)+H(S2/S1)\leq H(S1)+H(S2)$.

Consequently we can draw two useful ideas for the activity of characterizing economic phenomena through indicator systems:

- Although an economic phenomenon, through its complexity it requires large systems of indicators, it is imposed, due to comparability reasons, the dimensional reduction of these selection and/or aggregation systems.
- Aggregation generally leads to a loss in information, presenting the additional disadvantage that there cannot be obtained "good" aggregations (the sense with this optimality will be revealed in the following section).

2. An Impossibility Theorem for Indicators Aggregation

The preoccupation to characterize various phenomena in the socio-economic domain through mathematic models is already a constant practice in the scientific activities in the last few decades.

In general, for the numerical expression of economic phenomena we have at our disposal, a set of indicators, in the sense of the prior definition, that taken separately say very little about the complex analyzed phenomenon; there intervenes the salvaging idea of building the so-called aggregated indicators that include in their meaning the entire quantity of information offered separately by the component elements, while offering a image concerning the studied phenomenon better than the one given by each individual indicator.

The first steps in this area were made by Arrow, with his theorem about the impossibility of aggregating preferences in the social domain. Gh. Paun (1983) proves that an aggregated indicator sensitive and anti-catastrophic is compensatory[4], using elements of fuzzy sets theory. Next we will present the sense of the terms used by the theorem.[5]

Thus, we say that an aggregated indicator is *sensitive* if an increase of a component indicator, initially positive, does not lead to a decrease of the aggregated indicator, as a decrease of the initial indicator does not lead to an increase of the aggregated indicator.

For example, the income increase should not lead to the decrease of the life quality level, just as the decrease of the alphabetization degree should not lead to an increase in the human development index.

The discussion can also be carried in a negative way, thus, the increase of the cases of tuberculosis in a region should not cause the increase in life quality in that region,

Also, we say that an aggregated indicator is unexplosive (anti-catastrophic) if a "small" increase of an initial indicator does not lead to a "large" increase in the aggregated indicator. For example, the increase of the income with 10 lei must not lead to an exaggerated increase of the life quality.

An aggregated indicator is *non-compensatory* if a "large" modification of one of the component indicators is not accompanied by a inverse modification of another indicator in such a manner that the aggregated indicator equals with two completely different situations.

It is decent to observe that all social and economic indicators are both sensitive and un-explosive (or at least it would be ideal), consequently they must be compensatory.

In what follows, we will demonstrate this theorem, whose conclusion will be the same although with completely different hypotheses and with another formal approach of the terms implied.

We will assume that we have two indicators that action on a known set, finite and un-void, their values being real numbers (this hypothesis of quantitative and qualitative expression is essential).

Let A$\neq \emptyset$ the support set with card A $< \infty$; for example A can be the the set of all companies listed at the stock exchange, the set of economic agents in a region, the set of countries in the world or the set of regions in a geographical area.

We will consider also $i_k : A \to \mathbb{R}, k = 1,2$ two indicators defined on the support set to be submitted to the aggregation process[6]. We can also consider that the two indicators taken in consideration represent the price indexes for the only two companies of a stock exchange, to be aggregated into one stock exchange index.

In general, in statistics, aggregating an indicator that presents contents and different forms of expression is difficult, that is why we use standardized indicators, the new values being calculated with the relation: $x_j^{normat} = \dfrac{x_j - x_{\min}}{x_{\max} - x_{\min}} \in [0,1]$.

In this case we can restrict the indicators' codomain to the unit interval: $i_k : A \to [0,1], k = 1,2$.

We will then consider an aggregated indicator of the two primary indicators i_1 and i_2 as the function: $I_a \equiv f : \mathbb{R}^2 \to \mathbb{R}$, $I_a = f(i_1, i_2)$, for which we will state certain hypotheses.

In addition we will state that the aggregated indicator is compensatory if there is $\varepsilon > 0$ so that $f(i_1 + \varepsilon, i_2 - \varepsilon) = f(i_1, i_2)$.

Hypotheses for the aggregated indicator

H1. The aggregated indicator I_a is sensitive: if i_1 is a positive indicator(positive related to the aggregated indicator) then $\forall\ \varepsilon > 0$ we have $f(i_1 + \varepsilon, i_2) \geq f(i_1, i_2)$ and $f(i_1 - \varepsilon, i_2) \leq f(i_1, i_2)$.

As shown above, this hypothesis is natural (an increase of a positive primary indicator cannot lead to a decrease of the aggregated indicator).

H2. The aggregated indicator I_a has partial growths equally bounded:

$$\exists M > 0, \forall i_1, i_2 \in [0,1] \text{ and } \forall \varepsilon > 0 : \begin{cases} |f(i_1 + \varepsilon, i_2) - f(i_1, i_2)| \leq \varepsilon M \\ |f(i_1, i_2) - f(i_1, i_2 + \varepsilon)| \leq \varepsilon M \end{cases}$$

This hypothesis is an expression of the unexplosive aggregated indicator notion (a „small" increase of a primary indicator cannot lead to a "large" increase of the aggregated indicator). Indeed, a very small increase, infinitesimal, of any of the component indicator cannot lead, according to H2, to an explosive increase:

if $\varepsilon \to 0$ then $f(i_1+\varepsilon,i_2)-f(i_1,i_2)\to 0$ and $f(i_1,i_2)-f(i_1,i_2+\varepsilon)\to 0$.

In particular, if function *f* is partially continuous in relation to each argument, then H2 is verified. For example, the partial continuity in respect to the first variable involves an uniform continuity over the interval [0,1]:

$\forall\varepsilon>0$ and $\forall i_1\in[0,1]$, we have $i_1+\varepsilon-i_1<2\varepsilon$ so $\left|f(i_1+\varepsilon,i_2)-f(i_1,i_2)\right|\le\varepsilon$.

We can take in this case M=1.

Hypothesis H2 is also required by reality; the majority of the social-economic aggregated indicators, taking into account the way in which they are built, present the property of partial continuity, and implicitly H2 is verified.

Consequence:

The compensation function $g:\mathbb{R}\to\mathbb{R},\ g(x)=f(i_1+x,i_2-x)$ is continuous, i_1 and i_2 being considered constant.

Indeed, if i_1 and i_2 are constant, then the continuity of *g* function returns in fact to the partial continuty of function *f*.

The impossibility theorem for indicator optimal aggregation

In the conditions of hypotheses H1 and H2, there is $\varepsilon>0$ so that $f(i_1+\varepsilon,i_2-\varepsilon)=f(i_1,i_2)$ (the aggregated indicator is compensatory).

Demonstration

For x>0 we have the following:

$$\left|f(i_1+x,i_2-x)-f(i_1,i_2)\right|\le\left|f(i_1+x,i_2)-f(i_1,i_2)\right|+\left|f(i_1+x,i_2-x)-f(i_1+x,i_2)\right|\le$$
$$\le Mx+Mx=2Mx$$

(according to H1). Results that $\exists a>0$ so that $-a\le f(i_1+x,i_2-x)-f(i_1,i_2)\le a$, $\forall x\in(0,a)$.

Knowing that the compensation function $g(x)=f(i_1+x,i_2-x)$ is continuous, presents the Darboux property, then there is $\varepsilon>0$ so that $f(i_1+\varepsilon,i_2-\varepsilon)=f(i_1,i_2)$

(q.e.d.).

3. Observations

1. In proving the compensatory character, the sensitive aggregated indicator hypothesis did not intervene directly. Still, it is necessary in order to respect the relation between the model and reality.
2. In the case of aggregations with a number larger than 2 individual indicators, the idea being absolutely analogous, while the actual writing is somewhat harder.

A natural consequence of this theorem, with direct applicability in the study of capital markets, is that stock exchange indices, aggregated indicators, do not satisfy the optimality necessary conditions. In other words, there may be two situations on the market, apparently different, that are reflected by the same value of a stock exchange index.

A direct result concerns the comparability of stock exchange indices throughout time. Considering the conclusions of this theorem, we must show precaution when using stock exchange indices in comparisons.

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[2] Voineagu, V. (*et all*), **Analiza factoriala a fenomenelor social-economice in profil regional,** Ed. Aramis, Bucharest, 2002

[3] Ibidem.

[4] Paun, Gh. **An Impossibility Theorem for Indicators Aggregation,** Fuzzy Sets and Systems; Vol. 9, No. 2; #351, February 1983; pp. 205-210

[5] Paun, Gh. **Din spectacolul matematicii,** Ed. Albatros, Bucharest, 1983

[6] These can be, for example, the level of GDP per capita (reflecting economic development), the literacy rate (measuring the degree of education and culture of a population) or life expectancy at birth (reflecting the health status). These three indicators, through aggregation, contribute to defining the Human Development Index (HDI). See Isaic-Maniu, Al. (coordinator) **Dictionar de statistica generala,** Ed. Economica, Bucharest, 2003, p. 133