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A SIMPLE ANALYSIS OF CUSTOMERS IMPATIENCE IN MULTISERVER QUEUES

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Abstract: *Balking and reneging are two ways through which customer impatience finds reflection. In the analysis of stochastic reneging, it has traditionally been assumed that the distribution of patience time is state independent. However in many queuing systems, the customer is aware of its position in the system state. Hence in this paper, we assume that a customer who arrives at the queuing system gets to know the state of the system. Consequently, both balking and reneging are taken as function of system state. Both types of reneging are considered. Explicit closed form expressions of a number of performance measures are presented. A numerical example is presented to demonstrate the results derived rounds off the paper.*

Key words: *Balking, Impatience, Queuing, Reneging.*

1. Introduction

These days customers are busy entities. An assumption which is often attached to the analysis of many queuing model is the customers are willing to wait as long as it is necessary to obtain service. Our fast-paced life is often inconsistent with this assumption. In queuing terminology, two characteristics through which customer's impatience find reflection are balking and reneging. By balking, we mean the phenomenon of customers arriving for service into a non-empty queue and leaving without joining the queue. There is no balking from an empty queue. Haight (1957) has provided a rationale, which might influence a person to balk. It relates to the perception of the importance of being served which induces an opinion somewhere in between urgency, so that a queue of certain length will not be joined, to indifference where a non-zero queue is also joined.

A customer will be said to have reneged if after joining the system it gets impatient and leave the system without receiving service. On joining the system, it has a patience time such that in case service is unavailable within this patience time the customers renege.

Reneging can be of two types- reneging till beginning of service (henceforth referred to as R_BOS) and reneging till end of service (henceforth referred to as R_EOS). A customer can renege only as long as it is in the queue and we call this as reneging of type R_BOS. It cannot renege once it begins receiving service. A common example is the barbershop. A customer can renege while he is waiting in queue. However once service get started i.e. hair cut begins, the customer cannot leave till hair cutting is over. On the other hand, if customers can renege not only while waiting in queue but also while receiving service, we call such behavior as reneging of type R_EOS. An example is processing or merchandising of perishable goods.

In the analysis of reneging phenomena, one approach is to assume that each customer has a Markovian patience time, the distribution of which is state independent. However, it is our common day observation that there are systems where the customer is aware of its state in the system. For example customers queuing at the O.P.D. (out patient department) clinic of a hospital would know of their position in the queue. This invariably causes waiting customers to have higher rates of reneging in case their position in the queue is towards the end. It is not unreasonable then to expect that such customers who are positioned at a distance from the service facility have reneging rates, which are higher than reneging rates of customers who are near the service facility. In other words, we assume that customers are "state aware" and in this paper we model the reneging phenomenon in such a manner that the Markovian reneging rate is a function of the state of the customer in the system. Customers at higher states will be assumed to have higher reneging rates.

The subsequent sections of this paper are structured as follows. Section 2 contains a brief review of the literature. Section 3 and section 4 contains the derivation of steady state probabilities and performance measures respectively. We perform sensitivity analysis in section 5. A numerical example is discussed in section 6. Concluding statements are presented in section 7. The appendix presented in section 8 contains some derivation.

2. Literature Survey

One of the earliest work on reneging was by Barrer (1957) where he considered deterministic reneging with single server markovian arrival and service rates. Customers were selected randomly for service. In his subsequent work, Barrer (1957) also considered deterministic reneging (of both R_BOS and R_EOS type) in a multiserver scenario with FCFS discipline. The general method of solution was extended to two related queuing problems. Another early work on reneging was by Haight (1959). Ancher and Gafarian (1963) carried out an early work on markovian reneging with markovian arrival and service pattern. Ghosal (1963) considered a D/G/1 model with deterministic reneging. Gavish and Schweitzer (1977) also considered a deterministic reneging model with the additional assumption that arrivals can be labeled by their service requirement before joining the queue and arriving customers are admitted only if their waiting plus service time do not exceed some fixed amount. This assumption is met in communication systems. Kok and Tijms (1985) considered a single server queuing system where a customer becomes a lost customer when its service has not begun within a fixed time. Haghighi et al (1986) considered a markovian multiserver queuing model with balking as well as reneging. Each customer had a balking probability which was independent of the state of the system. Reneging discipline considered by them was R_BOS. Liu et al (1987) considered an infinite server markovian queuing system with reneging of type R_BOS. Customers had a choice of individual service or batch service, batch service being preferred by the customer. Brandt et al (1999) considered a S-server system with two FCFS queues, where the arrival rates at the queues and the service may depend on number of customers 'n' being in service or in the first queue, but the service rate was assumed to be constant for n>s. Customers in the first queue were assumed impatient customers with deterministic reneging. Boots and Tijms (1999) considered an M/M/C queue in which a customer leaves the system when its service has not begun within a fixed interval after its arrival. In this paper, they have given the probabilistic proof of 'loss probability', which was expressed in a simple formula involving the waiting time probabilities in the standard M/M/C queue. Wang et al (1999) considered the machine repair problem in which failed machines balk with probability (1-b) and renege according to a negative exponential distribution. Bae et al. (2001) considered an M/G/1 queue with deterministic reneging. They derived the complete formula of the limiting distribution of the virtual waiting time explicitly. Choi et al. (2001) introduced a simple approach for the analysis of the M/M/C queue with a single class of customers and constant patience time by finding simple markov process. Applying this approach, they analyzed the M/M/1 queue with two classes of customer in which class 1 customer have impatience of constant duration and class 2 customers have no impatience and lower priority than class 1 customers. Performance measures of both M/M/C and M/M/1 queues were discussed. Zhang et al. (2005) considered an M/M/1/N framework with markovian reneging where they derived the steady state probabilities and formulated a cost model. Some performance measures were also discussed. A numerical example was discussed to demonstrate how the various parameters of the cost model influence the optimal service rates of the system. El- Paoumy (2008) also derived the analytical solution of M^x/M/2/N queue for batch arrival system with markovian reneging. In this paper, the steady state probabilities and some performance measures of effectiveness were derived in explicit forms. Another paper on markovian reneging was by Yechiali and Altman (2008). They derived the probability generating function of number of customers present in the system and some performance measures were discussed. Choudhury (2009) considered a single server finite buffer queuing system (M/M/1/K) assuming reneging customers. Both rules of reneging were considered and various performance measures presented under both rules of reneging.

Other attempts at modeling reneging phenomenon include those by Baccelli et al (1984), Martin and Artalejo (1995), Shawky (1997), Choi, Kim and Zhu (2004), and Singh et al (2007), El- Sherbiny (2008) and El-Paoumy and Ismail (2009) etc.

An early work on balking was by Haight (1957). Another work using the concepts of balking and reneging in machine interference queue has been carried out by Al-Seedy and Al-Ibraheem (2001). There have been some papers in which both balking as well as reneging were considered. Here mention may be made of the work by Haghighi et al (1986), Shawky and El-Paoumy (2000), Zhang et al (2005), El- Paoumy (2008), El- Sherbiny (2008), Shawky and El-Paoumy (2008).

3. The Model and System State Probabilities

The model we deal with in this paper is the traditional M/M/k model with the restriction that customers may balk from a non-empty queue as well as may renege after they join the queue. We shall assume that the inter-arrival and service rates are λ and μ respectively. As for balking, we shall assume that each customer arriving at the system has a probability
'$1-p^{n-k+1}$' (for $n \geq k$ and 0 otherwise) of balking from a non-empty queue.

Customers joining the system are assumed to be of Markovian reneging type. We shall assume that on joining the system the customer is aware of its state in the system. Consequently, the reneging rate will be taken as a function of the customer's state in the system. In particular, a customer who is at state 'n' will be assumed to have random patience time following exp (ν_n). Under R_BOS, we shall assume that

$$\nu_n = \begin{cases} 0 & \text{for } n = 0,1,...,k \\ \nu + c^{n-k} & \text{for } n = k+1,... \end{cases}$$

and under R_EOS,

$$\nu_n = \begin{cases} 0 & \text{for n} = 0. \\ \nu & \text{for n} = 1,2,...,\text{k} \\ \nu + \text{c}^{\text{n-k}} & \text{for n} = \text{k}+1, \text{k}+2,.... \end{cases}$$

where c>1 is a constant.

Our aim behind this formulation is to ensure that higher the current state of a customer, higher is the reneging rate. Then it is clear that the constant c has to satisfy c>1. This reneging formulation also requires that as a customer progresses in the queue from state n (n≥k+2) to (n-1), the reneging distribution would shift from exp $(\nu+c^{n-k})$ to $(\nu+c^{n-k-1})$ under R_BOS. Similarly for R_EOS. In view of the memory less property, this shifting of reneging distribution is mathematically tractable, as we shall demonstrate in the subsequent sections. To the best of our knowledge, such a formulation of reneging distribution has not been attempted in literature. Advantages of the same are however obvious.

The steady state probabilities are derived by the Markov process method. We first analyze the case where customers renege only from the queue. Under R_BOS, let p_n denote the probability that there are 'n' customers in the system. The steady state probabilities under R_BOS are given below,

$$\lambda p_0 = \mu p_1 , \tag{3.1}$$

$$\lambda p_{n-1} + (n+1)\mu p_{n+1} = \lambda p_n + n\mu p_n \text{ ; n=1,2,...k-1,} \tag{3.2}$$

$$\lambda p^{n-k} p_{n-1} + \{k\mu+(n-k+1)\nu+c(c^{n-k+1}-1)/(c-1)\} p_{n+1} = \lambda p^{n-k+1} p_n + \{k\mu+(n-k)\nu+c(c^{n-k}-1)/(c-1)\} p_n ,$$

$$\text{n = k, k+1,...} \tag{3.3}$$

Solving recursively, we get (under R_BOS)

$$p_n = \{\lambda^n/(n!\mu^n)\} p_0 \text{ ;n=1,2,...k} \tag{3.4}$$

$$p_n = \left[\lambda^n p^{\{(n-k)(n-k+1)\}/2} \Big/ \{k!\mu^k \prod_{r=k+1}^{n} (k\mu + \overline{r-k}\nu + c\overline{c^{r-k}-1}/\overline{c-1})\} \right] p_0 \text{ ;n=k+1,} \tag{3.5}$$

where p_0 is obtained from the normalizing condition $\sum_{n=0}^{\infty} p_n = 1$ and is given as

$$p_0 = \left[\sum_{n=0}^{k} \lambda^n/(n!\mu^n) + \sum_{n=k+1}^{\infty} \lambda^n p^{\{(n-k)(n-k+1)\}/2} \Big/ \{k!\mu^k \prod_{r=k+1}^{n} (k\mu + \overline{r-k}\nu + c\overline{c^{r-k}-1}/\overline{c-1})\} \right]^{-1} \tag{3.6}$$

The steady state probabilities satisfy the recurrence relation. Under R_BOS

$$p_n = \{\lambda/(n\mu)\} p_{n-1} ; \quad \text{n=1,2,...,k}$$

and $p_n = \left[\lambda p^{n-k} / \{k\mu + \overline{n-k}\nu + c(c^{n-k}-1)/(c-1)\}\right] p_{n-1}$; n = k+1, k+2,...

We shall denote by K_{R_BOS} the probability that an arriving unit has to wait on arrival (under R_BOS). Then

$$K_{R\_BOS} = \Pr(N \geq k)$$

$$= \sum_{n=k}^{\infty} p_n . \tag{3.7}$$

We may call K_{R_BOS} as Erlang's second (Erlang's delay probability) formula for state dependent balking and state dependent reneging (R_BOS) in line with similar nomenclature in Medhi (2003, page 87).

Under R_EOS where customers may renege from queue as well as while being served, let q_n denote the probability that there are n customers in the system. Applying the Markov theory, we obtain the following set of steady state equations.

$$\lambda q_0 = (\mu + \nu) q_1 , \tag{3.8}$$

$$\lambda q_{n-1} + (n+1)(\mu+\nu)q_{n+1} = \lambda q_n + n(\mu+\nu)q_n \text{ ; n = 1,2,…,k-1} \quad (3.9)$$

$$\lambda p^{n-k} q_{n-1} + \{k\mu + (n+1)\nu + c(c^{n-k+1}-1)/(c-1)\}q_{n+1} = \lambda p^{n-k+1} q_n + \{k\mu + n\nu + c(c^{n-k}-1)/(c-1)\}q_n$$

; n = k, k+1, (3.10)

Solving recursively, we get (under R_EOS)

$$q_n = [\lambda^n / \{n!(\mu+\nu)^n\}]q_0; \quad \text{n = 1,2,…,k} \quad (3.11)$$

$$q_n = \left[\lambda^n p^{\{(n-k)(n-k+1)\}/2} \Big/ \{k!(\mu+\nu)^k \prod_{r=k+1}^{n} (k\mu + r\nu + c\overline{c^{r-k}-1}/\overline{c-1})\} \right] q_0 \text{ ;n = k+1,…}$$

(3.12)

where q_0 is obtained from the normalizing condition $\sum_{n=0}^{\infty} q_n = 1$ and is given as

$$q_0 = \left[\sum_{n=0}^{k} \lambda^n / \{n!(\mu+\nu)^n\} + \sum_{n=k+1}^{\infty} \lambda^n p^{\{(n-k)(n-k+1)\}/2} \Big/ \{k!(\mu+\nu)^k \prod_{r=k+1}^{n} (k\mu + r\nu + c\overline{c^{r-k}-1}/\overline{c-1})\} \right]^{-1}.$$

(3.13)

The recurrence relations under R_EOS are

$$q_n = [\lambda/\{n(\mu+\nu)\}]q_{n-1}; \quad \text{n = 1,2,…,k.}$$

and $q_n = \{\lambda p^{n-k} / (k\mu + n\nu + c\overline{c^{n-k}-1}/\overline{c-1})\}q_{n-1}$; n = k+1, k+2,…

We shall denote by K_{R_EOS} the probability that an arriving unit has to wait on arrival (under R_EOS). Then

$$K_{R\_EOS} = \Pr(N \geq k) = \sum_{n=k}^{\infty} q_n \quad (3.14)$$

which may be called Erlang's second (Erlang's delay probability) formula for state dependent balking and state dependent reneging (R_EOS).

4.Performance Measures

An important measure is 'L', which denotes the mean number of customers in the system. To obtain an expression for the same, we note that L=P′(1) where

$$P'(1) = \frac{d}{ds} P(s)|_{s=1}.$$

Here P(S) is the p.g.f. of the steady state probabilities. The derivation of P′(1) is given in the appendix. From (8.1.17) and (8.2.3), the mean system size under two reneging rules are

$$L_{R\_BOS} = (1/\nu)[\lambda(1-K_{R\_BOS}) + \{\lambda p_0 K_{R\_BOS}(p\lambda,\mu,\nu)\}/\{p^{k-1} p_0(p\lambda,\mu,\nu)\} - (\mu-\nu)\sum_{n=1}^{k} np_n - k(K_{R\_BOS} - p_k)(\mu-\nu)$$
$$-\{p_0 K_{R\_EOS}(c\lambda,\mu,\nu)\}/\{p_0(c\lambda,\mu,\nu)(c-1)c^{k-1}\} + (cK_{R\_BOS})/(c-1)].$$

(4.1)

$$L_{R\_EOS} = (1/\nu)[\lambda(1-K_{R\_EOS}) + \{\lambda q_0 K_{R\_EOS}(p\lambda,\mu,\nu)\}/\{p^{k-1} q_0(p\lambda,\mu,\nu)\} - \mu\sum_{n=1}^{k} nq_n - k\mu(K_{R\_EOS} - q_k)$$
$$-\{q_0 K_{R\_EOS}(c\lambda,\mu,\nu)\}/\{q_0(c\lambda,\mu,\nu)(c-1)c^{k-1}\} + (cK_{R\_EOS})/(c-1)].$$

(4.2)

Mean queue size can now be obtained and are given by

$$L_{q(R\_BOS)} = \sum_{n=k+1}^{\infty}(n-k)p_n$$

$$= L_{R\_BOS} - \sum_{n=1}^{k} np_n - k(K_{R\_BOS} - p_k)$$

$$=(1/\nu)[\lambda(1-K_{R\_BOS}) + \{\lambda p_0 K_{R\_BOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}p_0(p\lambda,\mu,\nu)\} - \mu\sum_{n=1}^{k} np_n - k\mu(K_{R\_BOS} - p_k)$$

$$-\{p_0 K_{R\_EOS}(c\lambda,\mu,\nu)\}\big/\{p_0(c\lambda,\mu,\nu)(c-1)c^{k-1}\} + (cK_{R\_BOS})/(c-1)].$$

$$L_{q(R\_EOS)} = L_{R\_EOS} - \sum_{n=1}^{k} nq_n - k(K_{R\_EOS} - q_k)$$

$$=(1/\nu)[\lambda(1-K_{R\_EOS}) + \{\lambda q_0 K_{R\_EOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}q_0(p\lambda,\mu,\nu)\} - (\mu+\nu)\sum_{n=1}^{k} nq_n - k(\mu+\nu)(K_{R\_EOS} - q_k)$$

$$-\{q_0 K_{R\_EOS}(c\lambda,\mu,\nu)\}\big/\{q_0(c\lambda,\mu,\nu)(c-1)c^{k-1}\} + (cK_{R\_EOS})/(c-1)].$$

Using Little's formula, one can calculate the average waiting time in the system and average waiting time in queue from the above mean lengths both under R_BOS and R_EOS.

Customers arrive into the system at the rate λ. However all the customers who arrive do not join the system because of balking. The effective arrival rate into the system is thus different from the overall arrival rate and is given by

$$\lambda^{e}{}_{(R\_BOS)} = \lambda\sum_{n=0}^{k-1} p_n + \sum_{n=k}^{\infty} p^{n-k+1} p_n$$

$$= \lambda(1-K_{R\_BOS}) + \{\lambda p_0 K_{R\_BOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}p_0(p\lambda,\mu,\nu)\}.$$

Similarly in case of R_EOS

$$\lambda^{e}{}_{(R\_EOS)} = \lambda(1-K_{R\_EOS}) + \{\lambda q_0 K_{R\_EOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}q_0(p\lambda,\mu,\nu)\}.$$

We have assumed that each customer has a random patience time following exp(ν). Clearly then, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rate under two reneging rules are given by

$$Avgrr_{(R\_BOS)} = \sum_{n=k+1}^{\infty}\{(n-k)\nu + c(c^{n-k}-1)\big/(c-1)\}\,p_n$$

$$=\nu\left\{p'(1) - \sum_{n=1}^{k} np_n\right\} - \nu k\left\{1-\sum_{n=0}^{k} p_n\right\} + \{1\big/(c-1)c^{k-1}\}\sum_{n=k+1}^{\infty} c^n p_n - \{c/(c-1)\}\left\{1-\sum_{n=0}^{k} p_n\right\}$$

$$= \lambda(1-K_{R\_BOS}) + \{\lambda p_0 K_{R\_BOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}p_0(p\lambda,\mu,\nu)\} - \mu\sum_{n=1}^{k} np_n - k\mu(K_{R\_BOS} - p_k).$$

$$Avg\,rr_{(R\_EOS)} = \sum_{n=1}^{k} n\nu q_n + \sum_{n=k+1}^{\infty}\{n\nu + c(c^{k-1}-1)\big/(c-1)\}q_n$$

$$=\nu Q'(1) + \{1\big/(c-1)c^{k-1}\}\sum_{n=k+1}^{\infty} c^n p_n - \{c/(c-1)\}\left\{1-\sum_{n=0}^{k} p_n\right\}$$

$$= \lambda(1-K_{R\_EOS}) + \{\lambda q_0 K_{R\_EOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}q_0(p\lambda,\mu,\nu)\} - \mu\sum_{n=1}^{k} nq_n - k\mu(K_{R\_EOS} - q_k).$$

In a real life situation, customers who balk or renege represent the business lost. Customers are lost to the system in two ways, due to balking and due to reneging. Management would like to know the proportion of total customers lost in order to have an idea of total business lost.

Hence the mean rate at which customers are lost (under R_BOS) is

$$\lambda - \lambda^{e}{}_{(R\_BOS)} + avgrr_{(R\_BOS)}$$

$$= \lambda - \mu \sum_{n=1}^{k} np_n - k\mu(K_{R\_BOS} - p_k).$$

and the mean rate at which customers are lost (under R_EOS) is

$$\lambda - \lambda^{e}{}_{(R\_EOS)} + avgrr_{(R\_EOS)}$$

$$= \lambda - \mu \sum_{n=1}^{k} nq_n - k\mu(K_{R\_EOS} - q_k).$$

These rates helps in the determination of proportion of customers lost which is of interest to the system manager as also an important measure of business lost. This proportion (under R_BOS) is given by

$$\{\lambda - \lambda^{e}{}_{(R\_BOS)} + avgrr_{(R\_BOS)}\}/\lambda$$

$$= 1 - (1/\lambda)[\mu \sum_{n=1}^{k} np_n + k\mu(K_{R\_BOS} - p_k)].$$

and the proportion (under R_EOS) is given by

$$\{\lambda - \lambda^{e}{}_{(R\_EOS)} + avgrr_{(R\_EOS)}\}/\lambda$$

$$= 1 - (1/\lambda)[\mu \sum_{n=1}^{k} nq_n + k\mu(K_{R\_EOS} - q_k)].$$

The proportion of customer completing receipt of service can now be easily determined from the above proportion.

The customers who leave the system from the queue do not receive service. Consequently, only those customers who reach the service station constitute the actual load of the server. From the server's point of view, this provides a measure of the amount of work he has to do. Let us call the rate at which customers reach the service station as λ^s .Then under R_BOS

$\lambda^s{}_{(R\_BOS)} = \lambda^e{}_{(R\_BOS)}$(1-proportion of customers lost due to reneging out of those joining the system)

$$= \lambda^{e}{}_{(R\_BOS)} \left\{ 1 - \sum_{n=k+1}^{\infty} (n-k)\nu p_n / \lambda^{e}{}_{(R\_BOS)} \right\}$$

$$= \lambda^{e}{}_{(R\_BOS)} - avgrr_{(R\_BOS)}$$

$$= \mu \sum_{n=1}^{k} np_n + k\mu(K_{R\_BOS} - p_k).$$

In case of R_EOS, one needs to recall that customers may renege even while being served and only those customers who renege from the queue will not constitute any work for the server. Thus

$\lambda^s{}_{(R\_EOS)} = \lambda^e{}_{(R\_EOS)}$(1-proportion of customers lost due to reneging from the queue out of those joining the system)

$$= \lambda^{e}{}_{(R\_EOS)} \left\{ 1 - \sum_{n=k+1}^{\infty} (n-k)\nu q_n / \lambda^{e}{}_{(R\_EOS)} \right\}$$

$$=\lambda^{e}{}_{(R\_EOS)}-\nu\left\{Q'(1)-\sum_{n=1}^{n}nq_n\right\}+k\nu\left(1-\sum_{n=0}^{k}q_n\right)$$

$$=\lambda^{e}{}_{(R\_EOS)}-\nu Q'(1)+\nu\sum_{n=1}^{n}nq_n+k\nu\left(K_{R\_EOS}-q_k\right)$$

$$=(\mu+\nu)\sum_{n=1}^{k}nq_n+k(\mu+\nu)(K_{R\_EOS}-q_k)+\{q_0K_{R\_EOS}(c\lambda,\mu,\nu)\}/\{q_0(c\lambda,\mu,\nu)(c-1)c^{k-1}\}-(cK_{R\_EOS})/(c-1).$$

In order to ensure that the system is in steady state, it is necessary for the rate of customers reaching the service station to be less than the system capacity. This translates to $(\lambda^s/k\mu)<1$.

5. Sensitivity Analysis.

It is interesting to examine and understand how server utilization varies in response to change in system parameters. The four system parameters of interest are λ,μ,ν. We place below the effect of change in these system parameters on server utilization. For this purpose, we shall follow the following notational convention in the rest of this section.

p_n (λ,μ,ν) and q_n (λ,μ,ν)will denote the probability that there are 'n' customers in a system with parameters λ,μ,ν in steady state under R_BOS and R_EOS respectively.

i) If $\lambda_1>\lambda_0$, then

$$\frac{p_0(\lambda_1,\mu,\nu)}{p_0(\lambda_0,\mu,\nu)}<1$$

$$\Rightarrow\frac{(\lambda_0-\lambda_1)}{\mu}+\ldots+\frac{(\lambda_0{}^k-\lambda_1{}^k)}{k!\,\mu^k}+\frac{p(\lambda_0{}^{k+1}-\lambda_1{}^{k+1})}{k!\,\mu^k(k\mu+\nu+c)}+\ldots<0$$

which is true. Hence $p_0\downarrow as\ \lambda\uparrow$.

ii) If $\mu_1>\mu_0$, then

$$\frac{p_0(\lambda,\mu_1,\nu)}{\mathrm{p}_0(\lambda,\mu_0,\nu)}>1$$

$$\Rightarrow\lambda\left(\frac{1}{\mu_0}-\frac{1}{\mu_1}\right)+\ldots+\frac{\lambda^k}{k!}\left(\frac{1}{\mu_0{}^k}-\frac{1}{\mu_1{}^k}\right)+\frac{\lambda^{k+1}p}{k!}\left\{\frac{1}{\mu_0{}^k(k\mu_0+\nu+c)}-\frac{1}{\mu_1{}^k(k\mu_1+\nu+c)}\right\}+\ldots>0$$

which is true. Hence $p_0\uparrow\quad as\ \mu\uparrow$.

iii) If $\nu_1>\nu_0$, then

$$\frac{p_0(\lambda,\mu,\nu_1)}{\mathrm{p}_0(\lambda,\mu,\nu_0)}>1$$

$$\Rightarrow\frac{\lambda^{k+1}p}{k!\mu^k}\left\{\frac{1}{(k\mu+\nu_0+c)}-\frac{1}{(k\mu+\nu_1+c)}\right\}$$

$$+\frac{\lambda^{k+2}p^3}{k!\mu^k}\left\{\frac{1}{(k\mu+\nu_0+c)\{k\mu+2\nu_0+c(c^2-1)/(c-1)\}}-\frac{1}{\{(k\mu+\nu_1+c)(k\mu+2\nu_2+c(c^2-1)/(c-1)\}}\right\}+\ldots..>0$$

which is true. Hence $p_0\uparrow as\ \nu\uparrow$.

The following can similarly be shown.

v) $q_0 \downarrow as\ \lambda \uparrow$

vi) $q_0 \uparrow as\ \mu \uparrow$

vii) $q_0 \uparrow as\ \nu \uparrow$

The managerial implications of the above results are obvious.

6. Numerical Example

To illustrate the use of our results, we apply them to a queuing problem. We quote below an example from Allen (2005, page 352).

'Customers arrive randomly (during the evening hours) at the Kitten house, the local house of questionable services, at an average rate of five per hour. Service time is exponential with a mean of 20 minutes per customer. There are two servers on duty.

So many queuing theory students visits the Kitten house to collect data for this book that proprietress, Kitty Callay (also known as the Cheshire Cat) make some changes. She trains her kittens to provide more exotic but still exponentially distributed service and add three more servers, for a total of five. Her captivated, customers still complain that the queue is too long. Kitty commissions her most favoured customer Gralre K. Renga to make a study of her establishment. He is to determine the…., the number of servers she should provide so that….
the probability that an arriving customers must wait for service will not exceed 0.25.'

This is a design problem. Here λ= 5/hr and μ= 3/hr. As required by the owner of the Kitten house, we examine the minimum number of servers with different choices of k. Though not explicitly mentioned, it is necessary to assume reneging and balking.

We shall assume that reneging distribution is state dependent following exp(ν_n) where ν_n is as described in section 3. Specifically, we shall assume ν=0.5/hr and considered the scenario with c=1.1. We further assume that balking rate is dependent of state and is 0.1.

Various performance measures of interest computed are given in the following table. These measures were arrived at using a FORTRAN 77 program coded by the authors. Different choices of k were considered. Results relevant with regard to the requirement that the Kitten house should provide servers so that the probability that an arriving customers will find all servers busy should be <0.25 are presented in the following table. (All rates in the table as per hour rates)

Table 1: Table of Performance Measures (with λ=5, μ=3, ν=0.5, p=0.9 and c=1.1)

Performance Measure	Number of servers		
	k=2	k=3	k=4
$\sum_{n=k+1}^{\infty} p_n$	0.50513	0.23437	0.08755
λ^s (i.e. arrival rate of customers reaching service station)	3.95868	4.62644	4.88363
Effective mean arrival rate(λ^e)	4.58476	4.82917	4.94089
Fraction of time server is idle (p_0)	0.18557	0.18879	0.18902
Average length of queue	0.37868	0.12356	0.03507
Average length of system	1.69824	1.66570	1.66295
Mean reneging rate	0.62608	0.20273	0.05726
Average balking rate	0.08305	0.03417	0.01182
Mean rate of customers lost	1.04132	0.37356	0.11637
Proportion of customers lost due to reneging, and balking.	0.20826	0.07471	0.02327

From the above table it is clear that an ideal choice of k could be k=3 with $\sum_{n=k+1}^{\infty} p_n$ =0.23437. It may be noted here that k=4 would also satisfy the design criteria. However, that would necessitate an additional server. Considering cost implications the idea would be to attain the design criteria with minimal number of severs. Under the assumption of balking and reneging, it appears that the proprietress need not increase the number of servers to five. Her design requirement would be met with three servers. She may therefore increase the number of servers by one.

7. Conclusion

The analysis of a multiserver Markovian queuing system with state-dependent balking and state dependent reneging has been presented. Even though balking and reneging have been discussed by others, explicit expression are not available. This paper makes a contribution here. Closed form expressions of number of performance measures have been derived. To study the change in the system corresponding to change in system parameters, sensitivity analysis has also been presented. A numerical example has been discussed to demonstrate results derived. The numerical example is of indicative nature meant to illustrate the benefits of our theoretical results in a design context.

8. Appendix

8.1. Derivation of P′(1) under R_BOS.

Let P(s) denote the probability generating function, defined by $P(s)=\sum_{n=0}^{\infty} p_n s^n$

From equation (3.2) we have

$$\lambda p_{n-1}+(n+1)\mu p_{n+1}=\lambda p_n+n\mu p_n ; \qquad n = 1,2\ldots k\text{-}1 .$$

Multiplying both sides of the equation by s^n and summing over n, we get

$$\lambda s\sum_{n=1}^{k-1} p_{n-1}s^{n-1}-\lambda\sum_{n=1}^{k-1} p_n s^n=\mu\sum_{n=1}^{k-1} np_n s^n-\frac{1}{s}\mu\sum_{n=1}^{k-1}(n+1)p_{n+1}s^{n+1} \quad (8.1.1)$$

From (3.3) we have

$$\lambda p^{n-k}p_{n-1}+\{k\mu+(n-k+1)\nu+c(c^{n-k+1}-1)/(c-1)\}p_{n+1}=\lambda p^{n-k+1}p_n+\{k\mu+(n-k)\nu+c(c^{n-k}-1)/(c-1)\}p_n$$

; n = k, k+1,…

Similarly multiplying both sides of the equation by s^n and summing over n

$$\lambda s\sum_{n=k}^{\infty} p^{n-k}p_{n-1}s^{n-1}-\lambda\sum_{n=k}^{\infty} p^{n-k+1}p_n s^n=\sum_{n=k}^{\infty}\{k\mu+(n-k)\nu+(c^{n-k+1}-c)/(c-1)\}p_n s^n$$
$$-\frac{1}{s}\sum_{n=k}^{\infty}\{k\mu+(n-k+1)\nu+(c^{n-k+2}-c)/(c-1)\}p_{n+1}s^{n+1} \quad (8.1.2)$$

Adding (8.1.1) and (8.1.2)

$$\Rightarrow \lambda s\left[\sum_{n=1}^{k-1}p_{n-1}s^{n-1}+\sum_{n=k}^{\infty}p^{n-k}p_{n-1}s^{n-1}\right]-\lambda\left[\sum_{n=1}^{k-1}p_n s^n+\sum_{n=k}^{\infty}p^{n-k+1}p_n s^n\right]$$
$$=\mu\sum_{n=1}^{k-1}np_n s^n+\sum_{n=k}^{\infty}\{k\mu+(n-k)\nu+(c^{n-k+1}-c)/(c-1)\}p_n s^n-\frac{1}{s}\left[\mu\sum_{n=1}^{k-1}(n+1)p_{n+1}s^{n+1}+\sum_{n=k}^{\infty}\{k\mu+(n-k+1)\nu+(c^{n-k+2}-c)/(c-1)\}p_{n+1}s^{n+1}\right]$$

$$\Rightarrow \lambda s\left[\left\{P(s)-\sum_{n=k}^{\infty} p_n s^n\right\}+p p_k s^k+p^2 p_{k+1} s^{k+1}+\ldots\right]-\lambda\left\{P(s)-p_0-\sum_{n=k}^{\infty} p_n s^n\right\}-\lambda p p_k s^k-\lambda p^2 p_{k+1} s^{k+1}-\ldots.$$

$$=\mu s\left\{P'(s)-\sum_{n=k+1}^{\infty} n p_n s^{n-1}\right\}+k\mu\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}+\nu s\{(k+1)p_{k+1}s^k+\ldots\}-\nu k\{p_{k+1}s^{k+1}+\ldots\}+\{1/(c-1)c^{k-1}\}\{c^k p_k s^k+c^{k+1}p_{k+1}s^{k+1}+\ldots\}$$

$$-\{c/(c-1)\}\left\{P(s)-\sum_{n=0}^{k-1} p_n s^n\right\}-\frac{1}{s}\left[\begin{array}{l}\mu s\left\{P'(s)-p_1-\sum_{n=k+1}^{\infty} n p_n s^{n-1}\right\}+k\mu\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}+\nu s\{(k+1)p_{k+1}s^k+\ldots\}-k\nu\{p_{k+1}s^{k+1}+\ldots.\}\\ +\{1/(c-1)c^{k-1}\}\{c^{k+1}p_{k+1}s^{k+1}+\ldots\}-\{c/(c-1)\}\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}\end{array}\right]$$

$$\Rightarrow \lambda s\left\{P(s)-\sum_{n=k}^{\infty} p_n s^n\right\}-\lambda\left\{P(s)-p_0-\sum_{n=k}^{\infty} p_n s^n\right\}+[\{\lambda(s-1)\}/p^{k-1}]\{p^k p_k s^k+p^{k+1}p_{k+1}s^{k+1}+\ldots\}$$

$$=\mu s\left\{P'(s)-\sum_{n=k+1}^{\infty} n p_n s^{n-1}\right\}+k\mu\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}+\nu s\left\{P'(s)-\sum_{n=1}^{k} n p_n s^{n-1}\right\}-k\nu\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}$$

$$+\{1/(c-1)c^{k-1}\}\left\{P(cs)-\sum_{n=0}^{k} p_n (cs)^n\right\}-\{c/(c-1)\}\left\{P(s)-\sum_{n=0}^{k-1} p_n s^n\right\}-\mu\left\{P'(s)-p_1-\sum_{n=k+1}^{\infty} n p_n s^{n-1}\right\}$$

$$-\frac{k\mu}{s}\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}-\nu\left\{P'(s)-\sum_{n=1}^{k} n p_n s^{n-1}\right\}+\frac{k\nu}{s}\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}-$$

$$\{1/(c-1)c^{k-1}s\}\left\{P(cs)-\sum_{n=0}^{k} p_n (cs)^n\right\}+\{c/(c-1)s\}\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}$$

$$\Rightarrow \lambda s\left\{P(s)-\sum_{n=k}^{\infty} p_n s^n\right\}-\lambda\left\{P(s)-p_0-\sum_{n=k}^{\infty} p_n s^n\right\}+[\{\lambda(s-1)\}/p^{k-1}]\{P(ps)-\sum_{n=0}^{k-1} p_n (ps)^n\}$$

$$=\mu s\left\{P'(s)-\sum_{n=k+1}^{\infty} n p_n s^{n-1}\right\}+k\mu\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}+\nu s\left\{P'(s)-\sum_{n=1}^{k} n p_n s^{n-1}\right\}-k\nu\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}$$

$$+\{1/(c-1)c^{k-1}\}\left\{P(cs)-\sum_{n=0}^{k} p_n (cs)^n\right\}-\{c/(c-1)\}\left\{P(s)-\sum_{n=0}^{k-1} p_n s^n\right\}-\mu\left\{P'(s)-p_1-\sum_{n=k+1}^{\infty} n p_n s^{n-1}\right\}$$

$$-\frac{k\mu}{s}\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}-\nu\left\{P'(s)-\sum_{n=1}^{k} n p_n s^{n-1}\right\}+\frac{k\nu}{s}\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}-$$

$$\{1/(c-1)c^{k-1}s\}\left\{P(cs)-\sum_{n=0}^{k} p_n (cs)^n\right\}+\{c/(c-1)s\}\left\{P(s)-\sum_{n=0}^{k} p_n s^n\right\}$$

$$\Rightarrow \lambda sP(s)-\lambda s\sum_{n=k}^{\infty}p_n s^n-\lambda P(s)+\lambda p_0+\lambda\sum_{n=k}^{\infty}p_n s^n+[\{\lambda(1-s)\}/p^{k-1}]P(ps)-[\{\lambda(1-s)\}/p^{k-1}]\sum_{n=0}^{k-1}p_n(ps)^n$$

$$=\mu sP'(s)-\mu s\sum_{n=k+1}^{\infty}np_n s^{n-1}+k\mu P(s)-k\mu\sum_{n=0}^{k}p_n s^n+\nu sP'(s)-\nu s\sum_{n=1}^{k}np_n s^{n-1}-k\nu P(s)+k\nu\sum_{n=0}^{k}p_n s^n$$

$$+\{1/(c-1)c^{k-1}\}P(cs)-\{1/(c-1)c^{k-1}\}\sum_{n=0}^{k}p_n(cs)^n-\{c/(c-1)\}P(s)+\{c/(c-1)\}\sum_{n=0}^{k-1}p_n s^n$$

$$-\mu P'(s)+\mu\frac{\lambda}{\mu}p_0+\mu\sum_{n=k+1}^{\infty}np_n s^{n-1}-\frac{k\mu}{s}P(s)+\frac{k\mu}{s}\sum_{n=0}^{k}p_n s^n-\nu P'(s)+\nu\sum_{n=1}^{k}np_n s^{n-1}+\frac{k\nu}{s}P(s)-\frac{k\nu}{s}\sum_{n=0}^{k}p_n s^n-$$

$$\{1/(c-1)c^{k-1}s\}P(cs)+\{1/(c-1)c^{k-1}s\}\sum_{n=0}^{k}p_n(cs)^n+\{c/(c-1)s\}P(s)-\{c/(c-1)s\}\sum_{n=0}^{k}p_n s^n$$

$$\Rightarrow P'(s)(\mu+\nu)=\lambda P(s)-\lambda\sum_{n=k}^{\infty}p_n s^n+\{\lambda P(ps)\}/p^{k-1}-\{\lambda\sum_{n=0}^{k-1}p_n s^n\}/p^{k-1}+\mu\sum_{n=k+1}^{\infty}np_n s^{n-1}-\frac{k\mu}{s}P(s)+\frac{k\mu}{s}\sum_{n=0}^{k}p_n s^n$$

$$+\nu\sum_{n=1}^{k}np_n s^{n-1}+\frac{k\nu}{s}P(s)-\frac{k\nu}{s}\sum_{n=0}^{k}p_n s^n-\frac{P(cs)}{(c-1)c^{k-1}s}+\frac{\sum_{n=0}^{k}p_n(cs)^n}{(c-1)c^{k-1}s}+\frac{cP(s)}{(c-1)s}-\frac{c\sum_{n=0}^{k}p_n s^n}{(c-1)s}$$

Now

$$\lim_{s\to 1-}P'(s)=\lim_{s\to 1-}\frac{1}{(\mu+\nu)}\left[\begin{array}{l}\lambda P(s)-\lambda\sum_{n=k}^{\infty}p_n s^n+\{\lambda P(ps)\}/p^{k-1}-\{\lambda\sum_{n=0}^{k-1}p_n s^n\}/p^{k-1}+\mu\sum_{n=k+1}^{\infty}np_n s^{n-1}-\frac{k\mu}{s}P(s)+\frac{k\mu}{s}\sum_{n=0}^{k}p_n s^n\\+\nu\sum_{n=1}^{k}np_n s^{n-1}+\frac{k\nu}{s}P(s)-\frac{k\nu}{s}\sum_{n=0}^{k}p_n s^n-\frac{P(cs)}{(c-1)c^{k-1}s}+\frac{\sum_{n=0}^{k}p_n(cs)^n}{(c-1)c^{k-1}s}+\frac{cP(s)}{(c-1)s}-\frac{c\sum_{n=0}^{k}p_n s^n}{(c-1)s}\end{array}\right]$$

$$P'(1)=\frac{1}{(\mu+\nu)}\left[\begin{array}{l}\lambda(1-\sum_{n=k}^{\infty}p_n)+(\lambda/p^{k-1})(P(p)-\sum_{n=0}^{k-1}p_n p^n)+\mu\{P'(1)-\sum_{n=1}^{k}np_n\}-k\mu(1-\sum_{n=0}^{k}p_n)\\+\nu\sum_{n=1}^{k}np_n+k\nu(1-\sum_{n=0}^{k}p_n)-\frac{P(c)}{(c-1)c^{k-1}}+\frac{\sum_{n=0}^{k}p_n c^n}{(c-1)c^{k-1}}+\frac{c}{(c-1)}(1-\sum_{n=0}^{k}p_n)\end{array}\right]$$

$$P'(1)=\frac{1}{\nu}\left[\begin{array}{l}\lambda(1-\sum_{n=k}^{\infty}p_n)+(\lambda/p^{k-1})(P(p)-\sum_{n=0}^{k-1}p_n p^n)-\mu\sum_{n=1}^{k}np_n-k\mu(1-\sum_{n=0}^{k}p_n)\\+\nu\sum_{n=1}^{k}np_n+k\nu(1-\sum_{n=0}^{k}p_n)-\frac{P(c)}{(c-1)c^{k-1}}+\frac{\sum_{n=0}^{k}p_n c^n}{(c-1)c^{k-1}}+\frac{c}{(c-1)}(1-\sum_{n=0}^{k}p_n)\end{array}\right]$$

(8.1.3)

Here $P(c)=\sum_{n=0}^{\infty}p_n(\lambda,\mu,\nu)c^n$ and $P(p)=\sum_{n=0}^{\infty}p_n(\lambda,\mu,\nu)p^n$ where the symbol $p_n(\lambda,\mu,\nu)$ is as described in section 7. We use p_n and $p_n(\lambda,\mu,\nu)$ interchangeably. However should any of the parameters λ,μ,ν change, it is explicitly stated. To obtain a closed form expression for P(c) and P(p), let us for the time being, consider two another queuing systems with parameters and assumptions similar to the queuing system we are presently considering except that the arrival rate is 'cλ' and 'pλ' respectively. For these new systems, the steady state equations are same as (4.1), (4.2) and (4.3) with 'λ' replaced by 'cλ' and 'pλ' respectively. Denoting the steady state probabilities of these new systems by $p_n(c\lambda,\mu,\nu)$ and $p_n(p\lambda,\mu,\nu)$ respectively, we can obtain under R_BOS

$$p_n(c\lambda,\mu,\nu)=\left\{(c\lambda)^n/n!\,\mu^n\right\}p_0(c\lambda,\mu,\nu)\text{; n=1,2,…,k} \qquad (8.1.4)$$

$$p_n(c\lambda,\mu,\nu)=\left[(c\lambda)^n p^{\{(n-k)(n-k+1)\}/2}\Big/\{k!\,\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}\right]p_0(c\lambda,\mu,\nu)$$

$$\text{; n = k+1,…} \qquad (8.1.5)$$

where

$$p_0(c\lambda,\mu,\nu)=\left[\sum_{n=0}^{k}(c\lambda)^n/(n!\,\mu^n)+\sum_{n=k+1}^{\infty}(c\lambda)^n p^{\{(n-k)(n-k+1)\}/2}\Big/\{k!\,\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}\right]^{-1}.$$

$$(8.1.6)$$

and $p_n(p\lambda,\mu,\nu)=\left\{(p\lambda)^n/n!\,\mu^n\right\}p_0(p\lambda,\mu,\nu)$; n=1,2,…,k-1

$$p_n(p\lambda,\mu,\nu)=\left[(p\lambda)^n p^{\{(n-k)(n-k+1)\}/2}\Big/\{k!\,\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}\right]p_0(p\lambda,\mu,\nu)$$

$$\text{n=k+1,…} \qquad (8.1.7)$$

where

$$p_0(p\lambda,\mu,\nu)=\left[\sum_{n=0}^{k}(p\lambda)^n/(n!\,\mu^n)+\sum_{n=k+1}^{\infty}(c\lambda)^n p^{\{(n-k)(n-k+1)\}/2}\Big/\{k!\,\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}\right]^{-1}.$$

$$(8.1.8)$$

Similarly, we can derive the steady state probabilities of these queuing systems under R_EOS.

Let $P(S;c\lambda,\mu,\nu)$ denotes the probability generating function of this new queuing system so that

$$P(S;p\lambda,\mu,\nu)=\sum_{n=0}^{\infty}p_n(p\lambda,\mu,\nu)s^n\ .$$

Now

$$P(p)=\sum_{n=0}^{\infty}p_n(\lambda,\mu,\nu)p^n$$

$$=p_0+\sum_{n=1}^{k}(p\lambda)^n p_0/n!\,\mu^n+\sum_{n=k+1}^{\infty}\left[(p\lambda)^n p^{\{(n-k)(n-k+1)\}/2}\Big/\{k!\,\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}\right]p_0$$

$$\Rightarrow\left[\{P(p)-p_0\}/p_0\right]=\sum_{n=1}^{k}(p\lambda)^n/n!\,\mu^n+\sum_{n=k+1}^{\infty}\left[(p\lambda)^n p^{\{(n-k)(n-k+1)\}/2}\Big/\{k!\,\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}\right]p_0$$

$$(8.1.9)$$

Now putting S=1 in $P(S;c\lambda,\mu,\nu)$ we get

$$P(1;p\lambda,\mu,\nu)=p_0(p\lambda,\mu,\nu)+\sum_{n=1}^{\infty}p_n(p\lambda,\mu,\nu)$$

$$\Rightarrow 1=p_0(p\lambda,\mu,\nu)+\sum_{n=1}^{k}(p\lambda)^n/n!\mu^n+$$

$$\sum_{n=k+1}^{\infty}\left[(p\lambda)^n p^{\{(n-k)(n-k+1)\}/2}\Big/\{k!\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}\right]p_0(p\lambda,\mu,\nu) \qquad \text{using}(8.1.7)$$

$$\Rightarrow 1=p_0(c\lambda,\mu,\nu)+\{(P(c)-p_0)/p_0\}p_0(c\lambda,\mu,\nu) \qquad \text{using}(8.1.9)$$

$$\Rightarrow P(p)=p_0/p_0(c\lambda,\mu,\nu).$$

Similarly under R_EOS,

$$Q(p)=q_0/q_0(p\lambda,\mu,\nu). \qquad (8.1.10)$$

Using the same procedure, we can obtain,
under R_BOS $P(c)=p_0/p_0(p\lambda,\mu,\nu)$ (8.1.11)
and under R_EOS $Q(c)=q_0/q_0(c\lambda,\mu,\nu)$. (8.1.12)

Again let $K_{R\_BOS}(p\lambda,\mu,\nu)=\sum_{n=k}^{\infty}p_n(p\lambda,\mu,\nu)$

$$=\sum_{n=k}^{\infty}[(p\lambda)^n p^{\{(n-k)(n-k+1)}\Big/\{k!\mu^k\prod_{r=k+1}^{n}(k\mu+\overline{r-k}\,\nu+c\overline{c^{r-k}-1}/\overline{c-1})\}]p_0(p\lambda,\mu,\nu)$$

$$=\sum_{n=k}^{\infty}p_n p^n\{p_0(c\lambda,\mu,\nu)/p_0\}.$$

Therefore,

$$\sum_{n=k}^{\infty}p_n p^n=p_0K_{R\_BOS}(p\lambda,\mu,\nu)/p_0(p\lambda,\mu,\nu) \qquad (8.1.13)$$

and under R_EOS we have

$$\sum_{n=k}^{\infty}q_n p^n=q_0K_{R\_BOS}(p\lambda,\mu,\nu)/q_0(p\lambda,\mu,\nu). \qquad (8.1.14)$$

similarly under R_BOS,

$$\sum_{n=k}^{\infty}p_n c^n=p_0K_{R\_BOS}(c\lambda,\mu,\nu)/p_0(c\lambda,\mu,\nu) \qquad (8.1.15)$$

and under R_EOS,

$$\sum_{n=k}^{\infty}q_n c^n=q_0K_{R\_BOS}(c\lambda,\mu,\nu)/q_0(c\lambda,\mu,\nu). \qquad (8.1.16)$$

Using these in (8.1.3), we

$$P'(1)=\frac{1}{\nu}\left[\begin{array}{l}\lambda(1-K_{R\_BOS})+\{\lambda p_0K_{R\_BOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}p_0(p\lambda,\mu,\nu)\}-(\mu-\nu)\sum_{n=1}^{k}np_n-k(\mu-\nu)(K_{R\_BOS}-p_k)\\-\{p_0K_{R\_BOS}(c\lambda,\mu,\nu)\}/\{(c-1)c^{k-1}p_0(c\lambda,\mu,\nu)\}+(cK_{R\_BOS})/(c-1)\end{array}\right].$$

{using (3.7)} (8.1.17)
where $p_0(c\lambda,\mu,\nu)$ and $p_0(p\lambda,\mu,\nu)$ are given in (8.1.6) and (8.1.8) respectively.

8.2. Derivation of Q′ (1) under R_EOS

From equation (3.9) we have,

$$\lambda q_{n-1} + (n+1)(\mu+\nu)q_{n+1} = \lambda q_n + n(\mu+\nu)q_n\,; \qquad n=1,2,\ldots,k-1.$$

Multiplying both sides of this equation by s^n and summing over n from we get

$$\lambda s\sum_{n=1}^{k-1} q_{n-1}s^{n-1} - \lambda\sum_{n=1}^{k-1} q_n s^n = (\mu+\nu)\sum_{n=1}^{k-1} nq_n s^n - \frac{1}{s}(\mu+\nu)\sum_{n=1}^{k-1}(n+1)q_{n+1}s^{n+1} \qquad (8.2.1)$$

From equation (3.10)

$$\lambda p^{n-k}q_{n-1} + \{k\mu + (n+1)\nu + c(c^{n-k+1}-1)/(c-1)\}q_{n+1} = \lambda p^{n-k+1}q_n + \{k\mu + n\nu + c(c^{n-k}-1)/(c-1)\}q_n$$

; n = k+1, k+2,....

Multiplying both sides of this equation by s^n and summing over n from we get

$$\lambda s\sum_{n=k+1}^{\infty} p^{n-k}q_{n-1}s^{n-1} - \lambda\sum_{n=k+1}^{\infty} p^{n-k+1}q_n s^n = \sum_{n=k+1}^{\infty}\{k\mu + n\nu + (c^{n-k+1}-c)/(c-1)\}q_n s^n$$
$$-\frac{1}{s}\sum_{n=k+1}^{\infty}\{k\mu + (n+1)\nu + (c^{n-k+2}-c)/(c-1)\}q_{n+1}s^{n+1}$$

(8.2.2)

Adding (8.2.1) and (8.2.2) and proceeding in a manner similar to section (8.1), we obtain,

$$Q'(1) = \frac{1}{\nu}\left[\begin{array}{l}\lambda(1-K_{R\_EOS}) + \{\lambda q_0 K_{R\_EOS}(p\lambda,\mu,\nu)\}/\{p^{k-1}q_0(p\lambda,\mu,\nu)\} - \mu\sum_{n=1}^{k} nq_n - k\mu(K_{R\_BOS}-q_k) \\ -\{q_0 K_{R\_EOS}(c\lambda,\mu,\nu)\}/\{(c-1)c^{k-1}q_0(c\lambda,\mu,\nu)\} + (cK_{R\_EOS})/(c-1)\end{array}\right]$$

.

{using (3.14), (8.1.14) and (8.1.15)} (8.2.3)

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