JOURNAL OF APPLIED QUANTITATIVE METHODS

International Symposium on Stochastic Models in Reliability Engineering, Life Sciences and Operations Management (SMRLO'10)

SEMI-MARKOV RELIABILITY MODEL OF THE COLD STANDBY SYSTEM

Franciszek GRABSKI
Prof, Department of Mathematics and Physics,
Polish Naval University,
Gdynia, Poland

E-mail: franciszekgr@onet.eu

Abstract: *The semi-Markov reliability model of the cold standby system with renewal is presented in the paper. The model is some modification of the model that was considered by Barlow & Proshan (1965), Brodi & Pogosian (1978). To describe the reliability evolution of the system, we construct a semi-Markov process by defining the states and the renewal kernel of that one. In our model the time to failure of the system is represented by a random variable that denotes the first passage time from the given state to the subset of states. Appropriate theorems from the semi-Markov processes theory allow us to calculate the reliability function and mean time to failure. As calculating an exact reliability function of the system by using Laplace transform is often complicated we apply a theorem which deals with perturbed semi-Markov processes to obtain an approximate reliability function of the system.*

***Key words**: semi-Markov process; perturbed process; reliability model; renewal standby system*

1. Description and Assumptions

We assume that the system consists of one operating series subsystem (unit), an identical stand-by subsystem and a switch (see Figure 1):

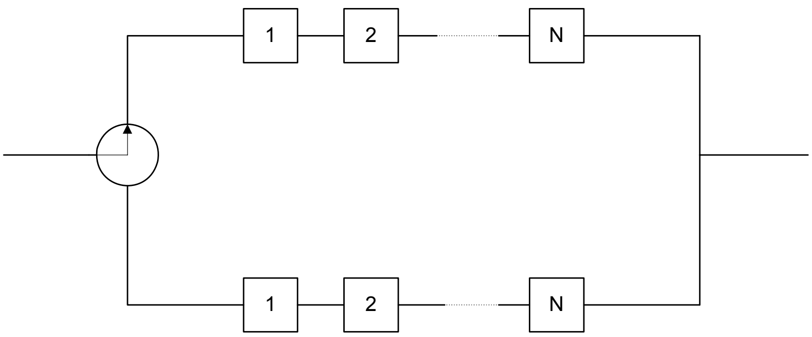

Figure 1. Diagram of the system

Each subsystem consists of N components. We assume that time to failure of those elements are represented by non-negative mutually independent random variables ζ_k, $k=1,...,N$, with distributions given by probability density functions $f_k(x)$, $x \geq 0$, $k=1,...,N$. When the operating subsystem fails, the spare is put in motion by the switch immediately. The failed subsystem is renewed. There is a single repair facility. A renewal time is a random variable having distribution depending on a failed component. We suppose that the lengths of repair periods of units are represented by identical copies of non-negative random variables γ_k, $k=1,...,N$, which have cumulative distribution functions $H_k(x)=P(\gamma_k \leq x)$, $x \geq 0$. The failure of the system occurs when the operating subsystem fails and the subsystem that has sooner failed in not still renewed or when the operating subsystem fails and the switch also fails. Let U be a random variable having binary distribution

$$b(k)=P(U=k)=a^k(1-a)^{1-k}, k=0,1,\ 0<a<1,$$

where $U=0$, if a switch is failed at the moment of the operating unit failure, and $U=1$, if the switch works at that moment. We suppose that the whole failed system is replaced by the new identical one. The replacing time is a non negative random variable η with CDF

$$K(x)=P(\eta \leqslant x),\ x \geqslant 0.$$

Moreover, we assume that all random variables mentioned above are independent.

2. Construction of Semi-Markov Reliability Model

To describe the reliability evolution of the system, we have to define the states and the renewal kernel. We introduce the following states:

0 – failure of the system;

k – renewal of the failed subsystem after a failure of k -th, $k=1,...,N$, component and the work of a spare unit

$N+1$ – both an operating unit and a spare are "up".

The scheme shown in Figure 2 presents functioning of the system. Let $0=\tau_0^*, \tau_1^*, \tau_2^*$- denote the instants of the states changes, and $\{Y(t): t \geq 0\}$ be a random process with the state space $S=\{0,1,...,N,N+1\}$, which keeps constant values on the half-intervals $[\tau_n^*, \tau_{n+1}^*), 0,1,...$, and is right-hand continuous. This process is not a semi-Markov one, as no memory property is satisfied for any instants of the state changes of that one.

Let us construct a new random process in a following way. Let $0=\tau_0$ and $\tau_1, \tau_2, ...$ denote instants of the subsystem failures or instants of the whole system renewal.

The random process $\{X(t): t \geq 0\}$ defined by equation

$$X(0)=0, \quad X(t)=Y(\tau_n) \quad \text{for} \quad t \in [\tau_n,\ \tau_{n+1}) \tag{1}$$

is the semi-Markov one.

To have a semi-Markov process as a model we have to define its initial distribution and all elements of its kernel. Recall that the semi-Markov kernel is the matrix of transition probabilities of the Markov renewal process

$$\boldsymbol{Q}(t) = [Q_{ij}(t) :\ i, j \in S], \tag{2}$$

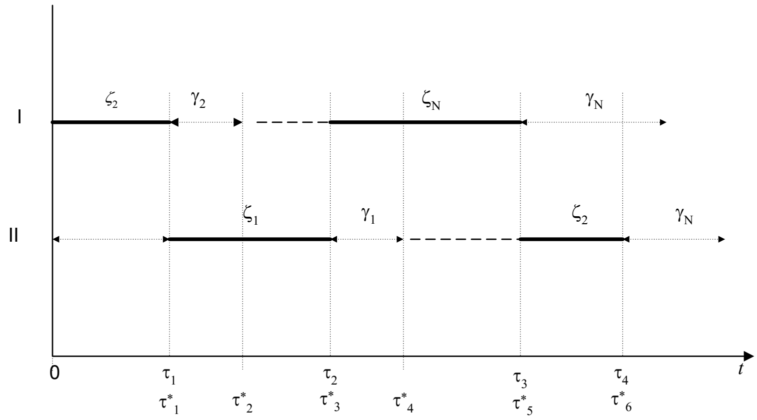

Figure 2. Reliability evolution of the standby system

where

$$Q_{ij}(t) = P(\tau_{n+1} - \tau_n \leqslant t,\ X(\tau_{n+1}) = j \,|\, X(\tau_n) = i), \quad t \geqslant 0. \tag{3}$$

From the definition of semi-Markov process it follows that the sequence $\{X(\tau_n) : n = 0,1,...\}$ is a homo-geneous Markov chain with transition probabilities

$$p_{ij} = P(X(\tau_{n+1}) = j \,|\, X(\tau_n) = i) = \lim_{t\to\infty} Q_{ij}(t). \tag{4}$$

The function

$$G_i(t) = P(\tau_{n+1} - \tau_n \leqslant t \,|\, X(\tau_n) = i) = \sum_{j\in S} Q_{ij}(t) \tag{5}$$

is a cumulative probability distribution of a random variable T_i that is called a waiting time of the state i. The waiting time T_i is the time spent in state i when the successor state is unknown. The function

$$F_{ij}(t) = P(\tau_{n+1} - \tau_n \leqslant t \,|\, X(\tau_n) = i,\ X(\tau_{n+1}) = j) = \frac{Q_{ij}(t)}{p_{ij}} \tag{6}$$

is a cumulative probability distribution of a random variable T_{ij} that is called a holding time of a state i, if the next state will be j. From here we have

$$Q_{ij}(t) = p_{ij} F_{ij}(t). \tag{7}$$

It follows from that a semi-Markov process with a discrete state space can be defined by the transition matrix of the embedded Markov chain: $P = [p_{ij} : i, j \in S]$ and a matrix of CDF of holding times: $F(t) = [F_{ij}(t) : i, j \in S]$.

In this case semi-Markov kernel has a form

$$\boldsymbol{Q}(t)=\begin{bmatrix} 0 & 0 & \cdots & 0 & Q_{0N+1}(t) \\ Q_{1\,0}(t) & Q_{1\,1}(t) & \cdots & Q_{1\,N}(t) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Q_{N\,0}(t) & Q_{N\,1}(t) & \cdots & Q_{N\,N}(t) & 0 \\ Q_{N+1\,0}(t) & Q_{N+1\,1}(t) & \cdots & Q_{N+1N}(t) & 0 \end{bmatrix}. \tag{8}$$

The semi-Markov $\{X(t): t \geq 0\}$ will be defined if we define all elements of the matrix $\boldsymbol{Q}(t)$.

For $j=1,\ldots,N$ we obtain

$$Q_{N+1\,j}(t)=P(X(\tau_{n+1})=j,\ \tau_{n+1}-\tau_n \leqslant t \mid X(\tau_n)=N+1)=$$

$$=P(A,\zeta_j \leqslant t,\zeta_i>\zeta_j \ \text{ for } \ i \neq j)=a\iint\limits_{D_{N+1\,j}}\ldots\int dF_1(x_1)dF(x_2)\cdots dF_N(x_N),$$

where

$$D_{N+1\,j}=\{(x_1,x_2,\ldots,x_N):\ 0\leqslant x_j\leqslant t, x_i>x_j,\ i\neq j\}.$$

Using Fubini theorem we obtain

$$Q_{N+1\,j}(t)=a\int_0^t\prod_{i\neq j}^{N}[1-F_i(x)]f_j(x)dx. \tag{9}$$

For $j=0$ we have

$$\begin{aligned} Q_{N+1\,0}(t)&=P(X(\tau_{n+1})=0,\ \tau_{n+1}-\tau_n\leqslant t \mid X(\tau_n)=N+1)= \\ &=P(A,\ \min(\zeta_1,\ldots,\zeta_N)\leq t)=(1-a)(1-\prod_{i=1}^{n}(1-F_i(t))). \end{aligned} \tag{10}$$

For $i,j=1,\ldots,N$ we get

$$Q_{ij}(t)=P(A,\zeta_j\leqslant t,\zeta_k>\zeta_j \text{ for } j\neq k, k=1,\ldots,N,\gamma_i<\zeta_j).$$

The same way we obtain

$$Q_{ij}(t)=a\int_0^t H_i(x)\prod_{k\neq j}^{N}[1-F_k(x)]f_j(x)dx. \tag{11}$$

For $i=1,\ldots,N$ and $j=0$ we have

$$\begin{aligned} Q_{i\,0}(t)&=P(\min(\zeta_1,\ldots,\zeta_N)\leqslant t,\ \gamma_i>\min(\zeta_1,\ldots,\zeta_N))+ \\ &+P(A',\ \min(\zeta_1,\ldots,\zeta_N)\leqslant t,\ \gamma_i<\min(\zeta_1,\ldots,\zeta_N))= \\ &=F(t)-a\int_0^t H_i(x)dF(x), \end{aligned} \tag{12}$$

where

$$F(x)=P(\min(\zeta_1,\ldots,\zeta_N)\leqslant t)=1-\prod_{k=1}^{N}[1-F_k(x)]. \tag{13}$$

From the assumption it follows that

$$Q_{0\,N+1}(t)=K(t). \tag{14}$$

All elements of the kernel $\boldsymbol{Q}(t)$ have been defined, hence the semi-Markov process $\{X(t): t \geq 0\}$ describing reliability of the renewal cold standby system has been constructed.

3. Exponential Time to Failure of Elements

Assuming the exponential time to failure of elements we obtain a special case of the model. Suppose that random variables ζ_k, $k = 1,\dots,N$ are exponentially distributed with parameters λ_k, $k = 1,\dots,N$, correspondingly. Hence

$$f_k(x) = \lambda_k e^{-\lambda_k x}, \quad x \geqslant 0.$$

Because of the no memory property of the exponential distribution, the assumption concerning of the whole subsystem renewal can be substituted by the assumption concerning failed element renewal.

In this case we obtain

$$Q_{N+1\,j}(t) = a \int_0^t \prod_{i \neq j}^{N} [e^{-\lambda_i x}] \lambda_j e^{-\lambda_j x} dx = a \frac{\lambda_j}{\Lambda}(1 - e^{-\Lambda t}), \quad t \geqslant 0 \tag{15}$$

for $j = 1,\dots,N$, where

$$\Lambda = \lambda_1 + \ldots + \lambda_N.$$

For $j = 0$ we obtain

$$Q_{N+1\,0}(t) = (1-a)(1 - e^{-\Lambda t}), \quad t \geqslant 0. \tag{16}$$

For $i, j = 1,\dots,N$

$$Q_{ij}(t) = a\lambda_j \int_0^t H_i(x) e^{-\Lambda x} dx. \tag{17}$$

For $j = 0$

$$Q_{i\,0}(t) = 1 - e^{-\Lambda t} - a\Lambda \int_0^t H_i(x) e^{-\Lambda x} dx. \tag{18}$$

4. Approximate Model

For simplicity we consider an approximate model. We can assume that the renewal time of the subsystem is a random variable γ having CDF

$$H(x) = \sum_{i=1}^{N} q_i\, H_i(x) \quad \text{where} \quad q_k = \frac{E(\gamma_k)}{\sum_{i=1}^{N} E(\gamma_i)}. \tag{19}$$

This way we obtain 3-state semi-Markov process with kernel

$$\boldsymbol{Q}(t) = \begin{bmatrix} 0 & 0 & Q_{02}(t) \\ Q_{10}(t) & Q_{11}(t) & 0 \\ Q_{20}(t) & Q_{21}(t) & 0 \end{bmatrix}, \tag{20}$$

where

$$Q_{02}(t) = K(t), \tag{21}$$

$$Q_{10}(t) = F(t) - a\int_0^t H(x)dF(x), \quad Q_{11}(t) = a\int_0^t H(x)dF(x), \tag{22}$$

$$Q_{20}(t) = (1-a)F(t), \quad Q_{21}(t) = aF(t).$$

Assume that, the initial state is 2. It means that an initial distribution is

$$\boldsymbol{p}(0) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{23}$$

Hence, the semi-Markov model has been constructed.

5. Reliability Characteristics

A value of a random variable

$$\Delta_A = \min\{n \in \mathbb{N}: \ X(\tau_n) \in A\} \tag{24}$$

denotes a discrete time (a number of state changes) of *a first arrival* at the set of states $A \subset S$ of the embedded Markov chain, $\{X(\tau_n): \ n \in \mathbb{N}_0\}$.

$$\Theta_A = \tau_{\Delta_A} \tag{25}$$

denotes a first passage time to the subset A or the time of a first arrival at the set of states $A \subset S$ of the semi-Markov process $\{X(t): t \geq 0\}$. A function

$$\Phi_{iA}(t) = P(\Theta_A \leqslant t \,|\, X(0) = i), \ t \geqslant 0 \tag{26}$$

is the Cumulative Distribution Function (CDF) of a random variable Θ_{iA} denoting the first passage time from the state $i \in A'$ to a subset A or the exit time of $\{X(t): t \geq 0\}$ from the subset A' with the initial state i. We will present some theorems concerning distributions and parameters of the random variables Θ_{iA} which are conclusions from theorems presented by Koroluk & Turbin (1976), Silvestrov (1980), Grabski (2002).

THEOREM 1

For the regular semi-Markov processes such that,

$$f_{iA} = P(\Delta_A < \infty | X(0) = i) = 1, \quad i \in A', \tag{27}$$

distributions $\Phi_{iA}(t) \quad i \in A'$ *are proper and they are the unique solutions of the equations system*

$$\Phi_{iA}(t) = \sum_{j\in A} Q_{ij}(t) \ + \ \sum_{k\in S} \int_0^t \Phi_{kA}(t-x)dQ_{ik}(x), \ \ i \in A'. \tag{28}$$

Applying a Laplace-Stieltjes (L-S) transformation for the system of integral equations we obtain the linear system of equations for (L-S) transforms

$$\tilde{\phi}_{iA}(s) = \sum_{j\in A} \tilde{q}_{ij}(s) + \sum_{k\in A'} \tilde{q}_{ik}(s)\tilde{\phi}_{kA}(s), \tag{29}$$

where

$$\tilde{\phi}_{iA}(s) = \int_0^\infty e^{-st} d\Phi_{iA}(t), \tag{30}$$

are L-S transforms of the unknown CDF of the random variables Θ_{iA}, $i \in A'$, and

$$\tilde{\phi}_{iA}(s) = \int_0^\infty e^{-st} d\Phi_{iA}(t), \tag{31}$$

are L-S transforms of the given functions $Q_{ij}(t),\ i,j \in S$. That linear system of equations is equivalent to the matrix equation

$$(\boldsymbol{I} - \tilde{\boldsymbol{q}}_{A'}(s))\,\tilde{\boldsymbol{\phi}}_{A'}(s) = \tilde{\boldsymbol{b}}(s), \tag{32}$$

where

$$\boldsymbol{I} = [\delta_{ij} :\ i,j \in A'] \tag{33}$$

is the unit matrix,

$$\tilde{\boldsymbol{q}}_{A'}(s) = [\tilde{q}_{ij}(s) :\ i,j \in A'] \tag{34}$$

is the square sub-matrix of the L-S transforms of the matrix $\tilde{\boldsymbol{q}}(s)$, while

$$\tilde{\boldsymbol{\phi}}_{A'}(s) = [\tilde{\phi}_{iA}(s) :\ i \in A']^T, \quad \tilde{\boldsymbol{b}}(s) = [\sum_{j \in A} \tilde{q}_{ij}(s) :\ i \in A']^T \tag{35}$$

are one column matrices of the corresponding L-S transforms.

The linear system of equations (29) for the L-S transforms allows us to obtain the linear system of equations for the moments of random variables $\Theta_{iA},\ i \in A'$.

THEOREM 2

If

- *assumptions of theorem 1 are satisfied,*
- $\bigvee_{d>0} \bigwedge_{i,j \in S} 0 < E(T_{ij}^2) \leqslant d,$
- $\bigwedge_{i \in A} \quad \mu_{iA}^2 = \sum_{n=1}^{\infty} n^2 f_{iA}(n) < \infty,$

then there exist expectations $E(\Theta_{iA}),\quad i \in A'$ *and second moments* $E(\Theta_{iA}^2), i \in A'$ *and they are unique solutions of the linear systems equations, which have following matrix forms*

$$(\boldsymbol{I} - \boldsymbol{P}_{A'})\overline{\boldsymbol{\Theta}}_{A'} = \overline{\boldsymbol{T}}_{A'}, \tag{36}$$

where

$$\boldsymbol{P}_{A'} = [p_{ij} :\ i,j \in A'],\ \overline{\boldsymbol{\Theta}}_{A'} = [E(\Theta_{iA}) :\ i \in A']^T,\ \overline{\boldsymbol{T}}_{A'} = [E(T_i) :\ i \in A']$$

$$(\boldsymbol{I} - \boldsymbol{P}_{A'})\overline{\boldsymbol{\Theta}^2}_{A'} = \boldsymbol{B}_A, \tag{37}$$

where

$$\boldsymbol{P}_{A'} = [p_{ij} : i,j \in A'], \quad \overline{\boldsymbol{\Theta}^2}_{A'} = [E(\Theta_{iA}^2) : i \in A']^T,$$

$$\boldsymbol{B}_A = [b_{iA} : i \in A']^T, \quad b_{iA} = E(T_i^2) + 2\sum_{k \in A'} p_{ik} E(T_{ik}) E(\Theta_{kA}),$$

and $\boldsymbol{I}$ *is the unit matrix.*

In our case the random variable Θ_{iA}, that denotes the first passage time from the state $i=2$ to the subset $A=\{0\}$ represents the time to failure of the system in our model. The function

$$R(t) = P(\Theta_{20} > t) = 1 - \Phi_{20}(t), \quad t \geqslant 0 \tag{38}$$

is the reliability function of the considered cold standby system with repair.

In this case the system of linear equations (29) for the Laplace-Stieltjes transforms with the unknown functions $\tilde{\phi}_{i0}(s), \quad t \geqslant 0, \quad i = 1,2$ is

$$\tilde{\phi}_{10}(s) = \tilde{q}_{10}(s) + \tilde{\phi}_{10}(s)\,\tilde{q}_{11}(s),\ \tilde{\phi}_{20}(s) = \tilde{q}_{20}(s) + \tilde{\phi}_{10}(s)\tilde{q}_{21}(s). \tag{39}$$

Hence

$$\tilde{\phi}_{10}(s) = \frac{\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)}, \quad \tilde{\phi}_{20}(s) = \tilde{q}_{20}(s) + \frac{\tilde{q}_{21}(s)\tilde{q}_{10}(s)}{1-\tilde{q}_{11}(s)}. \tag{40}$$

Consequently, we obtain the Laplace transform of the reliability function

$$\tilde{R}(s) = \frac{1-\tilde{\phi}_{20}(s)}{s}. \tag{41}$$

The transition matrix of the embedded Markov chain of the semi-Markov process $\{X(t): t \geq 0\}$ is

$$\boldsymbol{P} = \begin{bmatrix} 0 & 0 & 1 \\ p_{10} & p_{11} & 0 \\ p_{20} & p_{21} & 0 \end{bmatrix}, \tag{42}$$

where

$$p_{10} = 1 - p_{11}, \quad p_{11} = P(U = 1, \gamma < \zeta) = a \int_0^\infty H(x) dF(x),$$

$$p_{20} = 1 - a, \quad p_{21} = P(U = 1) = a.$$

The CDF of the waiting times $T_i, \quad i = 0, 1, 2$ are

$$G_0(t) = K(t), \quad G_1(t) = F(t), \quad G_2(t) = F(t).$$

Hence

$$E(T_0) = E(\eta), \quad E(T_1) = E(\zeta), \quad E(T_3) = E(\zeta). \tag{43}$$

In this case equation (37) takes the form of

$$\begin{bmatrix} 1-p_{11} & 0 \\ -a & 1 \end{bmatrix} \begin{bmatrix} E(\Theta_{10}) \\ E(\Theta_{20}) \end{bmatrix} = \begin{bmatrix} E(\zeta) \\ E(\zeta) \end{bmatrix}. \tag{44}$$

The solution of it is:

$$E(\Theta_{10}) = \frac{E(\zeta)}{1-p_{11}}, \quad E(\Theta_{20}) = E(\zeta) + \frac{a\,E(\zeta)}{1-p_{11}}. \tag{45}$$

6. An Approximate Reliability Function

In this case calculating an exact reliability function of the system by means of Laplace transform is a complicated matter. Finding an approximate reliability function of that system is possible by using results from the theory of semi-Markov processes perturbations. The perturbed semi-Markov processes are defined in different ways by different authors. We introduce Pavlov and Ushakov (1978) concept of the perturbed semi-Markov process presented by I.B. Gertsbakh (1984).

Let $A' = S - A$ be a finite subset of states and A be at least countable subset of S. Suppose $\{X(t): t \geq 0\}$ is SM process with the state space $S = A \cup A'$ and the kernel $\boldsymbol{Q}(t) = [Q_{ij}(t) : i, j \in S]$, the elements of which have the form $Q_{ij}(t) = p_{ij} F_{ij}(t)$.

Assume that

$$\varepsilon_i = \sum_{j \in A} p_{ij} \tag{46}$$

and

$$p_{ij}^0 = \frac{p_{ij}}{1-\varepsilon_i}, \quad i, j \in A'. \tag{47}$$

Let us notice that $\sum_{j \in A'} p_{ij}^0 = 1$.

A semi-Markov process $\{X(t) : t \geq 0\}$ with the discrete state space S defined by the renewal kernel $\boldsymbol{Q}(t) = [\, p_{ij} F_{ij}(t) : i, j \in S\,]$, is called the perturbed process with respect to SM process $\{X^0(t) : t \geqslant 0\}$ with the state space A' defined by the kernel $\boldsymbol{Q}^0(t) = [p_{ij}^0 F_{ij}(t) : i, j \in A']$.

We are going to present our version of theorem proved by I.B. Gertsbakh (1984).

The number

$$m_i^0 = \int_0^\infty [1 - G_i^0(t)] dt, \quad i \in A', \tag{48}$$

where

$$m_i^0 = \int_0^\infty [1 - G_i^0(t)] dt, \quad i \in A', \tag{49}$$

is the expected value of the waiting time in state i for the process $\{X^0(t) : t \geqslant 0\}$. Denote the stationary distribution of the embedded Markov chain in SM process $\{X^0(t) :\ t \geqslant 0\}$ by $\pi^0 = [\pi_i^0 :\ i \in A']$. Let

$$\varepsilon = \sum_{i \in A'} \pi_i^0 \varepsilon_i, \quad m^0 = \sum_{i \in A} \pi_i^0 m_i^0. \tag{50}$$

We are interested in the limiting distribution of the random variable $\Theta_{iA} = \inf\{t : X(t) \in A \,|\, X(0) = i\}, \quad i \in A'$, that denotes the first passage time from the state $i \in A'$ to the subset A.

THEOREM 3

If the embedded Markov chain defined by the matrix of transition probabilities $\boldsymbol{P} = [\, p_{ij} : i, j \in S]$ *satisfies the following conditions*

$$f_{iA} = P(\Delta_A < \infty | X(0) = i) = 1, \quad i \in A', \tag{51}$$

$$\bigvee_{c>0} \bigwedge_{i,j \in S} 0 < E(T_{ij}) \leqslant c, \tag{52}$$

$$\bigwedge_{i \in A} \mu_{iA} = \sum_{n=1}^\infty n f_{iA}(n) < \infty, \tag{53}$$

then

$$\lim_{\varepsilon \to 0} P(\varepsilon \Theta_{iA} \ > \ x) = e^{-\frac{x}{m^0}}, \tag{54}$$

where $\pi^0 = [\pi_i : i \in A']$ *is the unique solution of the linear system of equations*

$$\pi^0 = \pi^0 \boldsymbol{P}^0, \quad \pi^0 \boldsymbol{1} = 1. \tag{55}$$

From that theorem it follows that for small ε we get the following approximating formula

$$P(\Theta_{iA} > t) \ \approx \ \exp\left[- \frac{\sum_{i \in A'} \pi_i^0 \varepsilon_i}{\sum_{i \in A'} \pi_i^0 m_i^0} \, t \right], \quad t \geqslant 0. \tag{56}$$

The considered SM process $\{X(t):\ t \geqslant 0\}$ with the state space $S=\{0,1,2\}$ we can assume to be the perturbed process with respect to the SM process $\{X^0(t):\ t \geqslant 0\}$ with the state space $A'=\{1,2\}$ and the kernel

$$\boldsymbol{Q}^0(t)=\begin{bmatrix} Q^0_{11}(t) & 0 \\ Q^0_{21}(t) & 0 \end{bmatrix}, \tag{57}$$

where

$$Q^0_{11}(t)=p^0_{11}F_{11}(t), \quad Q^0_{21}(t)=p^0_{21}F_{21}(t).$$

Because $A=\{0\}$ and

$$\varepsilon_1=p_{10}=Q_{10}(\infty)=1-a\int_0^\infty G(x)f(x)dx,$$

then

$$p^0_{11}=\frac{p_{11}}{1-\varepsilon_1}=1.$$

From

$$\frac{Q_{11}(t)}{p_{11}}=F_{11}(t),$$

we get

$$Q^0_{11}(t)=F_{11}(t)=\frac{\int_0^t G(x)f(x)dx}{\int_0^\infty G(x)f(x)dx}.$$

Notice, that $\varepsilon_2=p_{20}=1-a$. Hence $p^0_{21}=\frac{p_{21}}{1-\varepsilon_2}=1$. Finally we obtain

$$Q^0_{21}(t)=F_{21}(t)=F(t).$$

The transition matrix of the embedded Markov chain of SM process $\{X^0(t):\ t \geqslant 0\}$ is

$$\boldsymbol{P}^0=\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}. \tag{58}$$

From the system of equations

$$[\pi^0_1,\pi^0_2]\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}=[\pi^0_1,\pi^0_2], \qquad \pi^0_1+\pi^0_2=1. \tag{59}$$

we get $\boldsymbol{\pi}^0=[1,\ 0]$. It follows from the theorem 3 that for a small ε

$$P(\Theta_{iA}>t)\approx\exp\left[-\frac{\sum_{i\in A'}\pi^0_i\varepsilon_i}{\sum_{i\in A'}\pi^0_i m^0_i}\,t\right], \quad t\geqslant 0, \tag{60}$$

where

$$m^0_i=\int_0^\infty[1-G^0_i(t)]dt,\ i\in A',\ G^0_i(t)=\sum_{j\in A'}Q^0_{ij}(t),\ \varepsilon=\sum_{i\in A'}\pi^0_i\varepsilon_i, \tag{61}$$

and

$$m^0 = \sum_{i \in A} \pi_i^0 m_i^0. \tag{62}$$

Therefore we have

$$\varepsilon = \varepsilon_1 = 1 - a \int_0^\infty H(x) f(x) dx,$$

$$m^0 = m_1^0 = \frac{\int_0^\infty x H(x) f(x) dx}{\int_0^\infty H(x) f(x) dx}.$$

For ε close to 0 we obtain the approximate reliability function of the system

$$R(t) = P(\Theta_{iA} > t) = P(\varepsilon \Theta_{iA} > \varepsilon t) \approx \exp\left[-\frac{\varepsilon}{m^0} t\right], \quad t \geqslant 0.$$

From a shape of the parameter ε it follows that we can apply this formula only if the number $P(\gamma \geqslant \zeta)$, denoting probability of a component failure during a period of an earlier failed component, is small.

Finally we obtain an approximate relation

$$R(t) = P\{\Theta_{20} > t\} \approx \exp\left[-\frac{c(1-ac)}{m_*} t\right], \tag{63}$$

where

$$c = \int_0^\infty H(x) f(x) dx = P(\gamma < \zeta),$$

$$m_* = \int_0^\infty x H(x) f(x) dx.$$

7. Conclusions

- The expectation $E(\Theta_{20})$ denoting the mean time to failure of the considered cold standby system is
  $$E(\Theta_{20}) = E(\varsigma) + \frac{a\, E(\varsigma)}{1 - p_{11}},$$
  where
  $p_{11} = a \int_0^\infty H(x) dF(x)$.
- The cold standby determines the increase of the mean time to failure
  $$1 + \frac{a}{1 - p_{11}}$$
  times.

- The approximate reliability function of the system is exponential with a parameter
  $$\Lambda = \frac{c(1-ac)}{m_*},$$
  where

$$c = P(\gamma < \zeta) = \int_0^\infty H(x) f(x) dx,$$

$$m_* = \int_0^\infty x H(x) f(x) dx.$$

References

1. Barlow, R.E. and Proshan, F. **Mathematical Theory of Reliability**, Wiley: New-York, London, Sydney, 1965
2. Brodi, S.M. and Pogosian, I.A. **Embedded Stochastic Processes in the Queue Theory**, Naukova Dumka, Kiev, 1978 (in Russian)
3. Gertsbakh, I.B. **Asymptotic methods in reliability theory: A review**, Adv. Appl. Prob., 16, 1984, pp. 147-175
4. Grabski, F.G. **Semi-Markov model of reliability and operation**, PAN IBS, Operation Research, 30, Warszawa, 2002, pp. 161 (in Polish)
5. Grabski, F.G. **Applications of Semi-Markov processes in safety and reliability analysis**, SSARS 2009, Gdańsk, 2009, pp. 94
6. Koroluk, W.S. and Turbin, A.F. **Semi-Markov Processes and Their Applications**, Naukova Dumka, Kiev, 1976 (in Russian)
7. Pavlov, I.V. and Ushakov, I.A. **The asymptotic distribution of the time until a semi-Markov process gets out of the kernel**, Engineering Cybernetics, 20 (3), 1978
8. Silvestrov, D.S. **Semi-Markov Processes with Discrete State Space**, Sovietskoe Radio, Moscow, 1980 (in Russian)