QUANTITATIVE RISK MANAGEMENT TECHNIQUES USING INTERVAL ANALYSIS, WITH APPLICATIONS TO FINANCE AND INSURANCE

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ABSTRACT

In this paper we study some risk management techniques using optimization problems under uncertainty. In decision making problems under uncertainty, the parameters of the models used can not be exactly described by real numbers, because of the imprecision of the data. In order to overcome this drawback the uncertainty of the parameters can be modeled by using interval numbers and interval random variables. Concepts of interval analysis are introduced in this article. Computational results are provided.

Keywords: *risk management, optimization, uncertainty, interval analysis*

1. INTRODUCTION

Interval analysis was introduced by Moore [4, 5]. The growing efficiency of interval analysis for solving various real life problems determined the extension of its concepts to the probabilistic case. Thus, the classical concept of random variable was extended to interval random variables, which has the ability to represent not only the randomness character, using the concepts of probability theory, but also imprecision and non-specifficity, using the concepts of interval analysis. The interval analysis based approach provides mathematical models and computational tools for modeling data and for solving optimization problems under uncertainty.

The occurrence of randomness in the model parameters can be formulated using stochastic programming models. Stochastic programming is widely used in many real decision making problems which arise in economy, engineering, social sciences and many other domains. It has been applied to a wide variety of fields such as manufacturing product and capacity planning, electrical generation capacity planning, financial planning and control, supply chain management, dairy farm expansion planning, macroeconomic modeling and planning, portfolio selection, transportation, telecommunications, banking and insurance. Recently a lot of papers investigate different methods for solving the portfolio selection problem [2,3, 6-10].

In this paper we study some quantitative risk management techniques based on interval analysis which can be used to model the uncertainty of data. The fundamental concepts of interval analysis are presented in Section 2. The theoretical basis provided is illustrated in Section 3 to model some financial data from Bucharest Stock Exchange. The results can be used to solve decisiona making problems under uncertainty. In Section 4 some conclusions are presented.

2. INTERVAL ANALYSIS

2.1. INTERVAL NUMBERS

Let x^L, x^U be real numbers, $x^L \le x^U$.

Definition 2.1. An *interval number* is a set defined by:

$X = \left\{ x \in \mathbf{R} | x^L \le x \le x^U;\ x^L, x^U \in \mathbf{R} \right\}$.

Remark 2.1. We will denote by $[x]$ the interval number $\left[x^L, x^U\right]$, with $x^L, x^U \in \mathbf{R}$.

We will denote by $\mathbf{IR}$ the set of all interval numbers.

Remark 2.2. If $x^L = x^U$, then the interval number $\left[x^L, x^U\right]$ is said to be a *degenerate* interval number. Otherwise, this is said to be a *proper* interval number.

Remark 2.3. A real number $x \in \mathbf{R}$ is equivalent with the interval number $[x, x]$.

Definition 2.2. An interval number $x = \left[x^L, x^U\right]$ is said to be:

- *negative*, if $x^U < 0$;
- *positive*, if $x^L > 0$;
- *nonnegative*, if $x^L \ge 0$;
- *nonpositive*, if $x^U \le 0$.

2.2. THE INTERVAL ARITHMETICS

Many relations and operations defined on sets or pairs of real numbers can be extended to operations on intervals.

Let $x = \left[x^L, x^U\right]$ and $y = \left[y^L, y^U\right]$ be interval numbers.

Definition 2.3. The *equality* between interval numbers is defined by:

$[x] = [y]$ if and only if $x^L = y^L$ and $x^U = y^U$.

Definition 2.4. The *maximum* between two interval numbers is defined by:

$\max\{[x],[y]\} = \left[\max\left\{x^L, y^L\right\}, \max\left\{x^U, y^U\right\}\right]$.

Definition 2.5. The *minimum* between two interval numbers is defined by:

$\min\{[x],[y]\} = \left[\min\left\{x^L, y^L\right\}, \min\left\{x^U, y^U\right\}\right]$.

Definition 2.6. The *intersection* between two interval numbers is defined by:

$[x] \cap [y] = \left[\max\left\{x^L, y^L\right\}, \min\left\{x^U, y^U\right\}\right]$.

Definition 2.7. The *union* between two interval numbers is defined by:

$$[x]\cup[y]=\begin{cases}\left[\min\{x^L,y^L\},\max\{x^U,y^U\}\right], & \text{if } [x]\cap[y]\neq\varnothing\\ \text{undefined}, & \text{otherwise}\end{cases}.$$

Definition 2.8. The *median* of the interval number $x=[x^L,x^U]$ is defined by:

$$m(x)=\frac{x^L+x^U}{2}.$$

Definition 2.9. The *length* of the interval number $x=[x^L,x^U]$ is defined by:

$$l(x)=x^U-x^L.$$

Definition 2.10. The *absolute value* of the interval number $x=[x^L,x^U]$ is defined by:

$$|x|=\{|y|:x^L\le y\le x^U\}.$$

Let $x=[x^L,x^U]$ an interval number and $a\in\mathbf{R}$ a real number.

Definition 2.11. The *product between the real number a and the interval number* $[x]$ is defined by:

$$a\cdot[x]=\{a\cdot x|x\in[x]\}=\begin{cases}[a\cdot x^L,a\cdot x^U], & \text{if } a>0\\ [a\cdot x^U,a\cdot x^L], & \text{if } a<0\\ [0], & \text{if } a=0\end{cases}.$$

Let "$\circ$" be one of the basic operations with real numbers, denoted by "$+$", "$-$", "$\cdot$", "$:$".

Definition 2.12. The *operation* $[x]\circ[y]$ is defined by:

$$[x]\circ[y]=\{x\circ y|x\in X,y\in Y\}.$$

The *summation* of two interval numbers defined by:

$$[x]+[y]=[x^L+y^L,x^U+y^U].$$

The *subtraction* of two interval numbers defined by:

$$[x]-[y]=[x^L-y^U,x^U-y^L].$$

2.3. INEQUALITIES BETWEEN INTERVAL NUMBERS

We will extend the classical inequality relations between real numbers to inequality relations between interval numbers.

Let $x=[x^L,x^U]$ and $y=[y^L,y^U]$ be interval numbers, with $x^L,x^U,y^L,y^U\in\overline{\mathbf{R}}$.

Definition 2.13. We say that:

1) $[x]<[y]$ if $x^U<y^L$;
2) $[x]\le[y]$ if $x^L<y^L$ and $x^U<y^U$;
3) $[x]\le_I[y]$ if $\begin{cases}x^L\le y^L\\ x^U<y^U\end{cases}$ or $\begin{cases}x^L<y^L\\ x^U\le y^U\end{cases}$ or $\begin{cases}x^L\le y^L\\ x^U\le y^U\end{cases}$;
4) $[x]\prec[y]$ if $x^L<y^U$

2.4. INTERVAL RANDOM VARIABLES

Let (Ω, K, P) be a probability space and $\mathbf{IR}$ be the set of all interval numbers.

Definition 2.14. An *interval random variable* is an application $[X]:\Omega \to \mathbf{IR}$,

$x(\omega) = [x^L(\omega), x^U(\omega)]$,

where $X^L, X^U : \Omega \to \mathbf{IR}$ are random variables, with $X^L \le X^U$ almost surely.

Definition 2.15. The interval random variable $[X]:\Omega \to \mathbf{IR}$ is said to be a *discrete interval random variable* if it takes values in a discrete subset of the real numbers.

The *probability distribution* of the discrete interval random variable $[X]:\Omega \to \mathbf{IR}$ can be expressed as:

$$[X]:\begin{pmatrix} [x_i] \\ p_i \end{pmatrix}_{i \in I},$$

where I is a discrete set of real numbers, or as:

$$[X]:\begin{pmatrix} [x_i^L, x_i^U] \\ p_i \end{pmatrix}_{i \in I}.$$

If $I = \{1, 2, ...n\}$ is a finite set, then the probability distribution of the discrete interval random variable $[X]:\Omega \to \mathbf{IR}$ can be expressed as:

$$[X]:\begin{pmatrix} [x_1^L, x_1^U] & [x_2^L, x_2^U] \ldots\ldots\ldots\ldots\ldots\ldots & [x_n^L, x_n^U] \\ p_1 & p_2 \ldots\ldots\ldots\ldots\ldots\ldots & p_n \end{pmatrix}.$$

Definition 2.16. The *probability mass function* of the discrete interval random variable $[X]:\Omega \to \mathbf{IR}$ is the function defined by

$$f:\mathbf{R} \to [0,1], \quad f(x) = \mathrm{P}([X] = [x]).$$

Definition 2.17. The *expectation* of the discrete interval random variable $[X]:\Omega \to \mathbf{IR}$ is the interval number defined by

$$\mathrm{E}([X]) = \sum_{i \in I} [x_i] p_i .$$

Definition 2.18. The *variance* of the discrete interval random variable $[X]:\Omega \to \mathbf{IR}$ is the interval number defined by

$$\mathrm{Var}([X]) = \sum_{i \in I} [x_i]^2 p_i - \left(\sum_{i \in I} [x_i] p_i \right)^2 .$$

3. CASE STUDY

In this section we will use the concepts defined in the previous section to model financial data. We have collected information concerning the values of the closing price ot a stock, TEL, listed at Bucharest Stock Exchange, between March 1, 2014 and June 25, 2014. The minimal value of the historical prices was 148400 and the maximal value was 228200, with an amplitude of 78800. **Table 1** presents the results obtained by organizing data into eight groups, with the amplitude of each group equal to 10000.

Table 1. The values of the historical prices

Interval	Absolute frequency
[148400; 158400)	50
[158400; 168400)	6
[168400; 178400)	8
[178400; 188400)	20
[188400; 198400)	16
[198400; 208400)	8
[208400; 218400)	4
[218400; 228400)	8

We model the data using a discrete interval random variable $[X]$, defined by:

$$[X]:\begin{pmatrix} [x_1^L, x_1^U] & [x_2^L, x_2^U] \ldots\ldots\ldots\ldots\ldots\ldots [x_8^L, x_8^U] \\ p_1 & p_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots p_8 \end{pmatrix},$$

with

$$[x_1^L, x_1^U] = [148400, 158400];\ p_1 = \frac{50}{120} = 0.41666;$$

$$[x_2^L, x_2^U] = [158400, 168400];\ p_2 = \frac{6}{120} = 0.05000;$$

$$[x_3^L, x_3^U] = [168400, 178400];\ p_3 = \frac{8}{120} = 0.06667;$$

$$[x_4^L, x_4^U] = [178400, 188400];\ p_4 = \frac{20}{120} = 0.16667;$$

$$[x_5^L, x_5^U] = [188400, 198400];\ p_5 = \frac{16}{120} = 0.13333;$$

$$[x_6^L, x_6^U] = [198400, 208400];\ p_6 = \frac{8}{120} = 0.06667;$$

$$[x_7^L, x_7^U] = [208400, 218400];\ p_7 = \frac{4}{120} = 0.03333;$$

$$[x_8^L, x_8^U] = [218400, 228400];\ p_8 = \frac{8}{120} = 0.06667.$$

$$[X]: 10^5 \cdot \begin{pmatrix} [1.484, 1.584] & [1.484, 1.584] & [1.484, 1.584] & [1.484, 1.584] & [1.484, 1.584] & [1.484, 1.584] & [1.484, 1.584] & [1.484, 1.584] \\ 0.41666 & 0.05000 & 0.06667 & 0.16667 & 0.13333 & 0.06667 & 0.03333 & 0.06667 \end{pmatrix}$$

The distribution of the discrete interval random variable can be used to compute relevant numerical values, such as expectation, standard deviation, skewness and kurtosis. The results obtained can be used to solve portfolio optimization problems.

4. CONCLUSIONS

The new approach based on interval analysis provides mathematical models and computational tools for modeling the imprecision of financial data and for solving decision making problems under uncertainty.

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