The use of the Premium calculation principles in actuarial pricing based scenario in a coherent risk measure

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Abstract
Calculation principles enable the development of the actuarial pricing process in the insurance sector both of life and of non-life. Among all principles assessed in this article we have chosen those verifying the so-called coherency criterion (Artzner, P. Delbaen, F. Eber, JM. Heath, D. (1999)), performing the theoretical-mathematic reasoning of such coherency criterion for all of them. Once those principles, more specifically two – the principle of net premium and the principle based on the distortion function in the form of power - are applied for the calculation of the risk premium both to a general and specific extent they will be applied to the Makeham Law for insurances with death cover: the whole life insurance.

Keywords: *Distortion function; coherent risk measure; Risk Premium; Insurance*

Introduction. Premium Calculation Principles based on Risk Measures

All companies assume risks that can potentially jeopardize their economic situation even forcing them into bankruptcy. The word risk goes normally hand in hand with luck, with uncertainty, so it is therefore related to the randomness of its occurrence and to the amount of the loss. It can be defined as the uncertainty regarding the onset of an event at a certain time and under specific conditions, generating therefore quantifiable loses. It is necessary to analyze the risks threatening life insurers with a view to optimize their management as risk assessment does not limit to their quantification (measurement) but also to achieve optimal protection against them and to try to avoid them. In this case we will focus on life insurances.

The main risks insurance companies have to deal with are mentioned in the following table. Among all of them, the death risk is the one used in this research paper, as it is the risk insurance companies have to face in insurances with death cover.

Table 1. Actuary risks

Market risk	**Observed in both areas, life and non-life**
Liquidity risk	Observed in both areas, life and non-life
Credit risk	Observed in both areas, life and non-life
Operational risk	Observed in both areas, life and non-life
Decrease in portfolio: - **Rescue** - **Reduction**	Specific for life insurance
Biometric risk: - **Mortality** - **Longevity**	Specific for life insurance

Source: Prepared by the author on the basis of Sandell, R (2003); Vegas Asensio, J (2000).

In order to implement an efficient risk management policy it will be necessary to previously quantify the risk using a tool for risk measurement. Therefore the next step is to define what a risk measure is. It is a functional (a function of functions) that assigns to a certain risk X a real non-negative number M(X) representing the additional amount that has to be added to the X loss in order to be acceptable by the insurance company (Gómez Déniz, E. Sarabia, JM.(2008)).

Thus, it will be rated (using a premium calculation principle) based on a risk measure as it adapts to the definition provided for this last one, as the premium assigns a real number to a random variable, which in the life insurance area represents the updated value of the product in question (for example, death insurance).

By definition, a premium calculation principle is a function H(X) assigning a real number to a risk X. Such real number is the premium. In practice, the premium calculation principle will depend on the distribution function F(X) which follows the random variable X, so instead of talking of a function H(X) we should talk about the functional H[F(X)] (Gerber, H. (1979)).

2. Coherent axiom for a premium calculation principle and its verification

The coherency criterion refers to the criterion granting economically rational contributions to the risk. Such coherency criterion has to provide correct information on financial assets, allowing its appropriate management (Tasche, D. (2000)).

Such coherency criterion is related to the fulfillment of four properties in a way that any premium calculation principle meeting such properties will be considered appropriate and optimal for a correct risk management, as it will perform an efficient allocation of the premium to the risk random variable (Artzner, P. Delbaen, F (1999); Landsman, Z. Sherris, M. (2001); Dhaene, J. Laeven, R. (2008)).

Table 2. Properties of a coherent risk measure

Positive coherency	M(aX)=aM(X). a≥0.
Invariance to translations	**M(X+a)=M(X)+a.**
Monotony	**Sea X1(w) and X2(w), with w ЄΩ, and where X1(w) ≤ X2(w), then M(X1)≤M(X2)**
Sub-additive	**M(X1+X2)≤M(X1)+M(X2)**

Source: Prepared by the author on the basis of Artzner, P. Delbaen, F (1999); Landsman, Z. Sherris, M. (2001); Dhaene, J. Laeven, R.Vanduffel, S. (2008).

Below we will mathematically develop for each and all of the existing premium calculation principles, the fulfillment of the four properties necessary to meet the coherency criterion. H(X) is the premium calculation principle.

2.1. Principle of the expected value and its particular case: the net premium principle

$H(X) = (1+\theta)E[X] \quad \theta > 0$, where θ is the surcharge factor.

This premium calculation principle shows a premium with an explicit surcharge.

(i) Sub-additive property:

Given any two risks X_1 and X_2 :

$$H(X_1+X_2)=H(X_1)+H(X_2)$$

$$H(X_1+X_2)=\left[(1+\theta)E(X_1)+(1+\theta)E(X_2)\right]=(1+\theta)\left[E(X_1)+E(X_2)\right]=$$

$$=(1+\theta)E(X_1)+(1+\theta)E(X_2)=H(X_1)+H(X_2)$$

Therefore this property is met.

(ii) Property of positive coherency:

Given a parameter $c \geq 0$ and a variable Y :

$$Y=cX$$

$$H(Y)=H(cX)=(1+\theta)E(cX)=c(1+\theta)E(X)=cH(X)$$

Therefore this property is met.

(iii) Property of monotony

Given any two risks X_1 and X_2, verifying that $X_1 \geq X_2$ and for $\theta > 0$:

$$E\left[X_1\right] \geq E\left[X_2\right]$$

$$H(X_1)=(1+\theta)E\left[X_1\right] \geq (1+\theta)E\left[X_2\right]=H(X_2)$$

$$H(X_1) \geq H(X_2)$$

Therefore the monotony property is met.

(iv) Property of invariance to translations

Given a parameter $c \geq 0$ and a variable $Y=c+X$:

Verifying that $H(Y)=H(X+c)=H(X)+c$

$$H(Y)=(1+\theta)E(c+X)=(1+\theta)\left[c+E(X)\right]=$$

$$=(1+\theta)c+(1+\theta)E(X)=(1+\theta)c+H(X)$$

As observed, it does not meet the property above mentioned.

This principle of the expected value does not meet the four desirable properties to be considered a coherency risk measure.

$\theta=0$: *Principle of net premium*

$$H(X)=(1+\theta)E\left[X\right]$$

$$\theta=0$$

$$H(X)=E(X)$$

(i) Property of sub-additive

Given any two risks X_1 and X_2 :

$$H(X_1+X_2)=E(X_1+X_2)=E(X_1)+E(X_2)=H(X_1)+H(X_2)$$

Therefore this property is met as the expectancy of the addition is the addition of expectancies.

(ii) Property of Positive coherency

Given a parameter $c \geq 0$ and a variable Y:

$$Y = cX;$$
$$H(Y) = H(cX) = E(cX) = cE(X)$$

Therefore the property is met.

(iii) Property of Monotony

Given any two risks X_1 and X_2, verifying that $X_1 \geq X_2$:

$$E[X_1] \geq E[X_2]$$
$$H(X_1) = E[X_1] \geq E[X_2] = H(X_2)$$
$$H(X_1) \geq H(X_2)$$

Therefore this property is met.

(iv) Property of Invariance to translations.

Given a parameter $c \geq 0$ and a variable $Y = c + X$:

$$H(X + c) = E(X + c) = E(X) + c = H(X) + c$$

The property abovementioned is met.

This net premium principle meets the four properties desirable to be considered a coherent risk measure.

2.2. Variance premium principle

$H(X) = E[X] + \alpha V[X]$ $\alpha \geq 0$ where α is the surcharge factor and $V[X]$ is the variance. This risk measure incorporates the safety surcharge factor in order to face random deviations of the random variable loses or loss rates. In this premium expression, the surcharge factor is proportional to the variance and shows an explicit surcharge of the premium.

(i) Property of sub-additive

Given any two risks X_1 and X_2:

$$H(X+Y) = E[X+Y] + \alpha V[X+Y] \nleq E[X] + E[Y] + \alpha [V[X] + V[Y] + 2Cov(X;Y)]$$

It does not meet the sub-additive principle unless both variables are independent.

(ii) Property of positive coherency

Given a parameter $c \geq 0$ and a variable Y:

$$Y = cX;$$
$$H(Y) = E[Y] + \alpha Var[Y] = E[cX] + \alpha V[cX] =$$
$$= cE[X] + \alpha c^2 V[X] = c[E[X] + \alpha c V[X]] \neq cH(X)$$

It does not meet the positive coherency principle.

(iii) Property of monotony

Given any two risks X_1 and X_2, verifying that $X_1 \geq X_2$:

$$E[X_1] \geq E[X_2]$$
$$H(X_1) = E[X_1] + \alpha V[X_1] \ngeq E[X_2] + \alpha V[X_2]$$

In this case it is not necessary to meet that the premium calculation principle of the first risk has to be higher or equal to the premium calculation principle of the second risk. Thus, it does not meet this property.

(iv) Property of invariance of translations.

Given a parameter $c \geq 0$ and the variable $Y = c + X$:

$$H(X+c) = E(X+c) + \alpha Var(X+c) = c + E(X) + \alpha Var(X) = c + H(X)$$

Therefore the principle of invariance to translations is met.

This premium calculation principle does not meet the four properties to be considered as a coherent risk measure.

2.3. Exponential premium

$H(X) = \frac{1}{\alpha} LogE\left[e^{\alpha X}\right] \quad \alpha \rangle 0$, where α is the so called Arrow-Pratt measure or absolute risk aversion associated to the person, showing an explicit surcharge of the premium.

(i) Property of invariance to translations

Given a parameter $c \geq 0$ and a variable $Y = c + X$:

$$H\left[x+c\right] = H\left[x\right] + c$$

$$H\left[X+c\right] = \frac{LogE\left[e^{\alpha(x+c)}\right]}{\alpha} = \frac{LogE\left[e^{\alpha x}e^{\alpha c}\right]}{\alpha} = \frac{Log\left[E\left(e^{\alpha x}\right)e^{\alpha c}\right]}{\alpha} =$$

$$= \frac{LogE\left[e^{\alpha x}\right] + Loge^{\alpha c}}{\alpha} = \frac{LogE\left[e^{\alpha x}\right]}{\alpha} + \frac{\alpha c}{\alpha} = H\left(x\right) + c$$

Therefore the invariance property to translations is met.

(ii) Property of positive coherency

Given a parameter $c \geq 0$ and a variable Y :

$$H(X) = \frac{LogE\left[e^{\alpha X}\right]}{\alpha}$$

$$H(Y) = H(cX) = cH(x)$$

$$H\left[cX\right] = \frac{LogE\left[e^{\alpha cX}\right]}{\alpha} \neq cH\left[X\right]$$

Therefore it does not meet the positive coherency property.

(ii) Property of Monotony

Given any two risks X_1 and X_2, verifying that $X_1 \leq X_2$:

$$H\left[X_1\right] = \frac{LogE\left[e^{\alpha(X_1+c)}\right]}{\alpha} = \frac{LogE\left[e^{\alpha X_1 + \alpha c}\right]}{\alpha}$$

$$H\left[X_2\right] = \frac{LogE\left[e^{\alpha(X_2+c)}\right]}{\alpha} = \frac{LogE\left[e^{\alpha X_2 + \alpha c}\right]}{\alpha}$$

$$H\left[X_1\right] = \frac{LogE\left[e^{\alpha(X_1+c)}\right]}{\alpha} \leq H\left[X_2\right] = \frac{LogE\left[e^{\alpha(X_2+c)}\right]}{\alpha}$$

Therefore this principle of monotony is met.

(iii) Property of sub-additive.

Given any two risks X_1 and X_2:

$$H(X_1) = \frac{\mathrm{LogE}\left[e^{\alpha X_1}\right]}{\alpha};$$

$$H(X_2) = \frac{\mathrm{LogE}\left[e^{\alpha X_2}\right]}{\alpha};$$

$$H(X_1+X_2) = \frac{\mathrm{LogE}\left[e^{\alpha(X_1+X_2)}\right]}{\alpha} = \frac{\mathrm{LogE}\left[e^{\alpha X_1}e^{\alpha X_2}\right]}{\alpha} \not\leq \frac{\mathrm{LogE}\left[e^{\alpha X_1}\right]}{\alpha} + \frac{\mathrm{LogE}\left[e^{\alpha X_2}\right]}{\alpha}$$

The principle of sub-additive is met only when both risks analysed are independent (Gómez Déniz, E. Sarabia, JM (2008)).

This principle of net premium does not meet the four properties desirable to be considered a coherent risk measure.

2.4. Esscher premium principle

$$H(X) = \frac{E\left[Xe^{\alpha X}\right]}{E\left[e^{\alpha X}\right]} \quad \alpha\rangle 0$$

This principle shows an explicit surcharge of the premium.

(i) Property of invariance to translations

Given a parameter $c \geq 0$ and a variable $Y = c + X$:

$$H(X+c) = \frac{E\left[(X+c)e^{\alpha(X+c)}\right]}{E\left[e^{\alpha(X+c}\right]} = \frac{E\left[(X+c)e^{\alpha(X+c)}\right]}{E\left[e^{\alpha X}e^{\alpha c}\right]} = \frac{E\left[Xe^{\alpha(X+c)} + ce^{\alpha(X+c)}\right]}{E\left[e^{\alpha X}\right]\left[e^{\alpha c}\right]} =$$

$$= \frac{E\left[Xe^{\alpha X}e^{\alpha c}\right] + cE\left[e^{\alpha X}e^{\alpha c}\right]}{e^{\alpha c}E\left[e^{\alpha X}\right]} = \frac{e^{\alpha c}E\left[Xe^{\alpha X}\right] + ce^{\alpha c}E\left[e^{\alpha X}\right]}{e^{\alpha c}E\left[e^{\alpha X}\right]} =$$

$$= \frac{e^{\alpha c}E\left[Xe^{\alpha X}\right]}{e^{\alpha c}E\left[e^{\alpha X}\right]} + \frac{ce^{\alpha c}E\left[e^{\alpha X}\right]}{e^{\alpha c}E\left[e^{\alpha X}\right]} = H(X) + c.$$

Therefore it meets the invariance property to translations.

(ii) Property of positive coherency

Given a parameter $c \geq 0$ and a variable $Y = cX$:

$$H(Xc) \neq cH(X)$$

$$\frac{E\left[cXe^{\alpha Xc}\right]}{e^{\alpha Xc}} \neq c\frac{E\left[Xe^{\alpha X}\right]}{e^{\alpha X}}$$

Therefore it does not meet the positive coherency principle.

(iii) Property of Monotony

Given any two risks X_1 and X_2, verifying that $X_1 \leq X_2$:

$$H(X_1) \leq H(X_2)$$

$$H(X_1) = \frac{E\left[(X_1 e^{\alpha X_1})\right]}{E\left[e^{\alpha X_1}\right]} \not\leq H(X_2) = \frac{E\left[(X_2 e^{\alpha X_2})\right]}{E\left[e^{\alpha X_2}\right]}$$

Where the first risk is lower or equal to the second risk, this property is not always met.

(iv) Property of sub-additive

Given any two risks X_1 and X_2:

$$H(X_1+X_2) \leq H(X_1)+H(X_2)$$

$$\frac{E\left[(X_1+X_2)\, e^{\alpha(X_1+X_2)}\right]}{E\left[e^{\alpha(X_1+X_2)}\right]} = \frac{E\left[(X_1\, e^{\alpha X_1}\, e^{\alpha X_2})+(X_2\, e^{\alpha X_1}\, e^{\alpha X_2})\right]}{E\left[e^{\alpha X_1}\, e^{\alpha X_2}\right]} \not\leq \frac{E\left[(X_1\, e^{\alpha X_1})\right]}{E\left[e^{\alpha X_1}\right]} + \frac{E\left[(X_2\, e^{\alpha X_2})\right]}{E\left[e^{\alpha X_2}\right]}$$

In this case the sub-additive property is not met.

This principle does not meet the four properties desirable to be considered as a coherent risk measure.

2.5. Wang distortion functional principle

$$H(X) = \int_0^\infty g\left(S_X(x)\right)dx = \int_0^\infty \left(S_X(x)\right)^{\frac{1}{\rho}} dx$$

This expression is the so-called Proportional Hazards Premiums Principle. Therefore, the distortion functional "g" is a tool used to build risk measures.

In cases where parameter ρ is valued as 1, the particular case of the risk measure takes place based on the principle of the net premium explained above.

It is worth highlighting that Wang, S (1995) demonstrated the four properties and therefore such demonstration is not going to be repeated. Regarding the last property, sub-additive, Wang, S (1995) demonstration for the case $\rho \geq 1$ is quite interesting. For $\rho \geq 1$ values, the premium calculation principle based on Wang distortion functional represents a coherent risk measure. And for the values of parameter $\rho \leq 1$ it also represents a coherent risk measure being the sub-additive property demonstrated by Hernández, M (2013). This principle is considered valid to be applied to within the life insurance scope as it verifies the coherence properties.

3. Application of the Net Premium Principle and the Principle of the Distortion Functional to Obtain the Single Risk Premium in a Death Insurance

The single risk premium is going to be expressed for insurances with death cover, the whole life insurance, on the grounds of the principles for the calculation of premiums considered as coherent risk measures (explained in section 2). In this insurance, the insurer undertakes to pay to the beneficiary of the policy the amount insured upon the death of the holder (Bowers, JR. Newton, L. Gerber, H. Jones, D. (1997)). In order to be entitled to the amount agreed, the holder will have to pay to the insurance company the amounts of the premiums, either on a periodic basis or by means of a single premium upon the date of the subscription of the insurance contract. In this case the random variable is the variable of residual life or the time to life from age x, T_x .

3.1. Calculation of the single risk premium with the net premium principle

The general expression of the premium for this type of insurance is (Bowers, JR. Newton, L. Gerber, H. Jones, D. (1997)).):

$$P = \int_0^\infty v^t d\left(G_x(t)\right)dt \,,$$

$$_t p_x = P\left(X > x + t / X > x\right) = 1 - G_x(t) = S_x(t)$$

$$P = \int_0^\infty v^t d(1 - S_x(t))dt = -\int_0^\infty v^t dS_x(t)$$

The following table shows single risk premiums for the law on survival used in this article. Their mathematical development can be seen in the doctoral dissertation of the author (Hernández, M. (2013)).

Table 3. Single risk premium by application of the Net premium principle

Single risk premium	Single risk premium. Makeham Law
$P = 1 - \int_0^1 S_x\left(\frac{\text{Ln}\, z}{\text{Ln}\, v}\right)dz$	$P = \frac{g^{C^x}\left(\text{LnS} + \text{Lnv} + C^{x+1}\text{Lng}\right) - \text{Lnv}}{g^{C^x}\left(\text{LnS} + \text{Lnv} + C^{x+1}\text{Lng}\right)}$

Source: Prepared by the author on the basis of the doctoral dissertation of the author.

3.2 Calculation of the single premium with the distortion functional principle

The distortion functional transforms the survival function through the operator g, based on the expression of the single risk premium calculated in section 3.1. It is expressed as power, being ρ parameter considered as the parameter of risk aversion (Tse, Y-K (2009)).

$$P = \int_0^\infty g\left(S_X(x)\right)dx = \int_0^\infty g\left(S_X(x)\right)^{\frac{1}{\rho}} dx$$

The following table shows the single risk premiums for each of the survival laws used in this article. Their mathematical development can be studies in the doctoral dissertation of the author (Hernández, M. (2013)).

Table 4. Single risk premium surcharged by application of the distortion functional principle

Single risk premium	Single risk premium. Makeham Law
$P_{rec} = 1 - \int_0^1 S_x\left(\frac{\text{Lnz}}{\text{Lnv}}\right)^{\frac{1}{\rho}}dz$	$P_{rec} = \frac{g^{C^x \frac{1}{\rho}}\left(\frac{1}{\rho}\text{LnS} + C^{x+1}\frac{1}{\rho}\text{Lng} + \text{Lnv}\right) - \text{Lnv}}{g^{C^x \frac{1}{\rho}}\left(\frac{1}{\rho}\text{LnS} + C^{x+1}\frac{1}{\rho}\text{Lng} + \text{Lnv}\right)}$

Source: Prepared by the author on the basis of the doctoral dissertation of the author

4. Conclusions

This article has analysed the main premium calculation principles within the actuarial area. Such analysis has consisted on the mathematical development, for each and all of them, of the properties defining coherence. Among all premium calculation principles chosen -the net premium principle, the expected value principle, the variance principle, the exponential premium principle, the Esscher premium principle as well as the distortion functional principle- only two of them verify the coherence axiom. The net premium principle is a particular case of the expected value and therefore represents a coherence risk measure. The disadvantage is that it provides a premium free of surcharges and therefore insurance companies have to work with outdated death tables to use death risk under the rates of the human group considered in such tables. In turn, and this is the main contribution of this research work, the principle of the distortion function has been applied to date within the scope of general insurances. In this study, it is applied for the first time to the calculation of the single risk premium in death insurances for the life insurance scope (Hernández, M (2013)). This is a principle that represents a coherent risk measure for parameter values $\rho \leq 1$, which are the values that have to be verified in the type of insurance used in order to obtain a premium higher than the net premium (that obtained by the first of the principles).

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